EXERCISES XVI (2)

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Problem (2)**.** (Burnside) Let $\chi \in Irr(G)$ be such that $\chi(1) > 1$. Show that $\chi(g) = 0$ for some $g \in G$.

Proof. First, $\sum_{g \neq 1} |\chi(g)|^2 = |G| - |\chi(1)|^2 < |G| - 1$ by SOR. Assume $\chi(g) \neq 0$ for all $g \in G$. Then, d defined in problem 1 cannot be zero and $|d| \geq 1$ since it is an integer. By Arithmetic-Geometric means,

$$\frac{\sum_{g \neq 1} |\chi(g)|^2}{|G| - 1} \geq \left(\prod_{g \neq 1} |\chi(g)|^2 \right)^{1/(|G|-1)} = \left(|d|^2\right)^{1/(|G|-1)} \geq 1$$

But since LHS of the inequality is less than 1, we have a contradiction, i.e. $\chi(g) = 0$ for some $g \in G$. □

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