# Assignment 15, Problem 5

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**Abstract**

5. Show that the Feit-Thompson Theorem ("Odd groups are solvable") is equivalent to the statement that finite nonabelian simple groups have even order.

FT implies nonabelian simple groups have even order:

First, we note that if $G$ is both simple and solvable, then $G = \mathbb{Z}/p\mathbb{Z}$ for some prime $p$. For if $G$ is solvable, there exists a composition series for $G$ in which all the factor groups cyclic groups of prime order. In particular, we have $G = H_n \triangleright H_{n-1}$ and $H_n/H_{n-1} = \mathbb{Z}/p\mathbb{Z}$. But since $G$ is also simple by assumption, the only normal subgroup it contains is $\{1\}$, so we conclude that $H_{n-1} = 1$ and $G = \mathbb{Z}/p\mathbb{Z}$.

Now, suppose that $G$ has odd order. By the Feit-Thompson Theorem, $G$ is solvable. If in addition $G$ is nonabelian, $G$ cannot be simple, because the only solvable simple groups are $\mathbb{Z}/p\mathbb{Z}$, and these are all abelian. So if $G$ is nonabelian and of odd order, $G$ is not simple. This is equivalent to the statement that if $G$ is nonabelian and simple, then $G$ has even order.

Nonabelian simple groups have even order implies FT:

Well, suppose that NSGHEO implies FT for all groups of order less than $|G|$, and that $G$ is a group of odd order. Then $G$ is either abelian, in which case it is clearly solvable, or $G$ is not simple, so there is a normal subgroup $N$ of $G$. By 1st grade, $|G|$ odd implies both $|N|$ and $|G/N|$ odd, and FT for groups of order less than $|G|$ implies that $N$, $G/N$ are both solvable. Since solvable groups are closed under extension, this implies that $G$ is solvable, ie. that FT holds for all groups of order less than and including $|G|$. Hence, by induction, NSGHEO implies FT.