Assignment 15, Problem 4

November 8, 2007

Abstract

4. Prove the following generalization of Frobenius' Integrality Theorem: For $\chi \in Irr(G), g \in G$, and $d = gcd(\chi(1), |g|)$, $\frac{d\chi(g)}{\chi(1)}$ is an algebraic integer.

Recall that Frobenius' Integrality Theorem states that $\frac{|g|\chi(g)}{\chi(1)}$ is an algebraic integer, so this is a stronger statement because the numerator is potentially smaller.

Nonetheless, we can prove this essentially just using Frobenius' original theorem. First of all, $\chi(g)$ is always an algebraic integer, so it is a root of some monic polynomial in $\mathbb{Z}[x]$, say $\widetilde{f}(x) = x^n + ... + a_0$. Then $|g|\chi(g)/\chi(1)$ is a root of $f(x) = x^n + |g|/\chi(1)a_{n-1}x^{n-1} + ... + (|g|/\chi(1))^n a_0$.

By FIT, we also know that $|g|\chi(g)/\chi(1)$ is an algebraic integer, so it is the root of some monic polynomial in $\mathbb{Z}[x]$, say $h(x) = x^r + ... + b_0$. By the same logic, $\chi(g) = \frac{\chi(1)}{|g|} * |g|\chi(g)/\chi(1)$ satisfies $\widetilde{h}(x) = x^r + \chi(1)/|g|b_{r-1}x^{r-1} + ... + (\chi(1)/|g|)^r b_0$. Since $\chi(g)$ is a root of both $\widetilde{h}$ and $\widetilde{f}$ and $\widetilde{f}$ is irreducible in $\mathbb{Q}[x]$ (irreducible in $\mathbb{Z}[x]$ implies irreducible in $\mathbb{Q}[x]$), $\widetilde{f}$ divides $\widetilde{h}$, or $n \leq r$. Conversely, $|g|\chi(g)/\chi(1)$ is a root of both f and h, and h is irreducible, so $r \leq n$. We conclude that $n = r$.

Then, h divides f, and they are both polynomials with the same degree, hence they differ by a constant multiple. That multiple is easily deduced by looking at the leading coefficients: so $f = h$! So we conclude that f was already a monic polynomial in $\mathbb{Z}[x]$, that is, $(|g|/\chi(1))^i a_{n-i}$ is an integer for all i. That is great, because then replacing $|g|$ by $d = gcd(\chi(1), |g|)$, we still have $(d/\chi(1))^i a_{n-i}$ is an integer for all i, hence $d\chi(g)/\chi(1)$ is a root of the monic integer equation $e(x) = x^n + d/\chi(1)a_{n-1}x^{n-1} + ... + (d/\chi(1))^n a_0$, ie. it is an algebraic integer, as desired!