

Assignment 15, Problem 2

November 8, 2007

Abstract

2. For $\chi, \chi_1, \dots, \chi_m \in \text{Irr}(G)$ (any G), show that TFAE: a) $\ker\chi_1 \cap \dots \cap \ker\chi_m \subseteq \ker\chi$. b) χ appears in some product $\chi_1^{i_1} \dots \chi_m^{i_m}$. Show that a), b) are implied by, but not equivalent to: c) $\chi \in \mathbb{Z}[\chi_1, \dots, \chi_m]$ (the subring of $\text{Ch}(G)$ generated by $\{\chi_i\}$).

a) \rightarrow b): Let $H = \ker\chi_1 \cap \dots \cap \ker\chi_m$. H is a normal subgroup of G , and we can consider all of these as representations of $K = G/H$, and denote the resulting representations as $\tilde{\chi}_i$. Now, for two distinct characters φ, ψ , $\ker\varphi \cap \ker\psi = \ker\varphi\psi$, because if D, E are the corresponding representations, $D(g) \otimes E(g)$ is the identity iff $D(g)$ and $E(g)$ are both the identity.

Based on this last observation, we conclude that $\ker(\tilde{\chi}_1 \dots \tilde{\chi}_m) = \{1\}$, or $\tilde{\chi}_1 \dots \tilde{\chi}_m$ is a faithful representation of K . Then by the Burnside-Brauer theorem, $\tilde{\chi}$ appears in some power $(\tilde{\chi}_1 \dots \tilde{\chi}_m)^r$. Now, since H is contained in the kernel of every character $\chi, \chi_1, \dots, \chi_m$, we did not lose any information in passing to $K = G/H$, that is, in fact χ appears in $(\chi_1 \dots \chi_m)^r$, as desired.

b) \rightarrow a): Note that $\ker\varphi \subseteq \ker\varphi^i$ for any character φ . This together with (*) above gives $\ker\chi_1 \cap \dots \cap \ker\chi_m \subseteq \ker\chi_1^{i_1} \dots \chi_m^{i_m}$. Another observation is that if χ appears in $\chi_1^{i_1} \dots \chi_m^{i_m}$, then $\ker\chi_1^{i_1} \dots \chi_m^{i_m} \subseteq \ker\chi$. This is easiest to see by considering the representation D for which χ is a character, and E for which $\chi_1^{i_1} \dots \chi_m^{i_m}$ is a character. Since χ appears in $\chi_1^{i_1} \dots \chi_m^{i_m}$, there is a third representation F for which the matrix $E(g)$ equals the block sum of $D(g)$ and $F(g)$. So if $E(g)$ is the identity, $D(g)$ must be, or if $g \in \ker\chi_1^{i_1} \dots \chi_m^{i_m}$, then $g \in \ker\chi$.

Hence we can finally conclude that $\ker\chi_1 \cap \dots \cap \ker\chi_m \subseteq \ker\chi_1^{i_1} \dots \chi_m^{i_m} \subseteq \ker\chi$, as desired.

c) \rightarrow b): If $\chi \in \mathbb{Z}[\chi_1, \dots, \chi_m]$, then χ can be expressed as some polynomial in χ_1, \dots, χ_m . In particular, χ must appear (not necessarily alone!) as a summand of some monomial in χ_1, \dots, χ_m , which is another way of saying b).

b) \nrightarrow c): Part b) of Ex. 1 provides an example where b) holds, but not c). That is, $\chi_2 = (1, 1, \zeta, \zeta^2)$ (ζ a primitive cube root of unity) appears in φ^2 , but χ_2 itself is not in $\mathbb{Z}[\varphi]$!