Assignment 15, Problem 1

November 8, 2007

Abstract

1. a) Let $\varphi = (2,0,1) \in \mathrm{Irr(G)}$ where $G = S_3$. Show that the character ring Ch(G) (consisting of all virtual characters) is equal to $\mathbb{Z}[\varphi]$ (the ring generated by φ). b) Let $\varphi = (3,-1,0,0) \in \mathrm{Irr(G)}$ where $G = A_4$. Show that Ch(G) is *not* equal to $\mathbb{Z}[\varphi]$.

a) Well, $\mathbb{Z}[\varphi]$ is clearly a subring of Ch(G), because $\varphi \in \mathrm{Ch(G)}$. So it just remains to show that $\mathrm{Ch(G)} \in \mathbb{Z}[\varphi]$, and to see this it suffices to show that the generators χ_1, χ_2, χ_3 of $Ch(G)$ are in $\mathbb{Z}[\varphi]$.

Well, φ is the character of a faithful representation, so as we showed in class a few weeks ago, every irreducible representation will appear in some power of φ. The question is just whether it will be possible to separate it out with coefficient 1. So we compute $\varphi^2 = (4,0,1)$, and $\langle \varphi^2, \chi_1 \rangle = \langle \varphi^2, \chi_2 \rangle = \langle \varphi^2, \chi_1 \rangle = 1$. That is, $\varphi^2 = \chi_1 + \chi_2 + \chi_3$. But $\chi_1 = \varphi^0$, $\chi_3 = \varphi$, and so $\chi_2 = \varphi^2 - \varphi - \varphi^2$. Hence, Ch(G) is a subring of $\mathbb{Z}[\varphi]$ and hence the two rings are equal, as desired.

b) Again φ is a faithful character, so every irreducible character of G appears in some power of φ. However, the point here is that unlike in the previous case, some characters always appear in powers of φ together with the same coefficient, and hence never appear alone in $\mathbb{Z}[\varphi]$. In particular, every vector in $\mathbb{Z}[\varphi]$ has real coefficients, so the irreducible character $\chi_2 = (1,1,\zeta,\zeta^2)$ (ζ a primitive cube root of unity) never appears in $\mathbb{Z}[\varphi]$, so $\mathbb{Z}[\varphi] \neq \mathrm{Ch(G)}$. (That is, in $\mathbb{Z}[\varphi]$, χ_2 and χ_3 always appear with the same coefficient.)