Math 110 Exam 2 Name:________________________

Summer 2008 SID:________________________

Each part of each problem is worth the number of points stated in parentheses. You must show all work to get any partial credit, which will be awarded for certain progress in a problem only if no substantially false statements have been written. If you are using a theorem, please write "By a theorem..," and if you remember the name of the theorem, please write "By the (—–) Theorem...," etc. If you are using a homework problem, please write "By a homework problem...," and if you remember which homework problem, please write "By homework problem $x.y.z$," etc. You may not say that a certain problem follows from a theorem/homework problem if the theorem/homework problem is exactly what you are being asked to prove (for example, problem 5.1 on this exam).

FOR INSTRUCTOR'S USE ONLY:

		Points
Definitions	1	
	2	
	3	
	4	
Theorems	1	
	2	
	3	
	4	
Examples	1	
	2	

		Points
	3	
	4	
True/False	1	
	2	
	3	
	4	
Problems	1	
	2	
	3	
	4	

1 Definitions

Define the following terms/symbols. You may state results without proof in this section.

Problem 1.1. (*2 points*) Positive Operator.

Problem 1.2. (*2 points*) Characteristic Polynomial.

Problem 1.3. (*2 points*) Generalized Eigenvector.

Problem 1.4. (*4 points*) Jordan Canonical Form.

2 Theorems

State the following theorems without proof.

Problem 2.1. (*2 points*) Rank-Nullity.

Problem 2.2. (*2 points*) Cayley-Hamilton.

Problem 2.3. (*2 points*) Riesz Representation.

Problem 2.4. (*4 points*) Primary Decomposition.

3 Examples

Find an example of the following. You must show that they are, in fact, examples!

Problem 3.1. (*2 points*) An irreducible polynomial of degree 2.

Problem 3.2. (*2 points*) An operator without a Jordan canonical form.

Problem 3.3. (*2 points*) A self adjoint unitary that is not $\pm I$.

Problem 3.4 (Systems of Matrix Units 1)**.** (*4 points*) A system of matrix units for $L(H)$ of size m is a collection of nonzero partial isometries $\{V_{i,j} | i, j \in [m]\} \subset L(H)$ such that

(i) $V_{i,j}^* = V_{j,i}$ for all $i, j \in [m]$,

(ii) $V_{i,j}V_{k,l} = \delta_{j,k}V_{i,l} = \begin{cases} 0 & \text{if } j \neq k \\ V_{i,l} & \text{fi } j = k \end{cases}$ for all $i, j, k, l \in [m]$, and

(iii) $\sum_{i=1}^{m} V_{i,i} = I$.

Find systems of matrix units of size 2 and 4 in $M_4(\mathbb{F}) = L(\mathbb{F}^4)$. For simplicity, we will identify $A \in M_4(\mathbb{F})$ with $L_A \in L(\mathbb{F}^4)$.

4 True/False

Determine (prove) whether the following statements are true or false.

Problem 4.1. (*2 points*) Every matrix in $M_n(\mathbb{C})$ is similar to its adjoint.

Problem 4.2. (*2 points*) The matrices $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ are similar.

Problem 4.3. (*2 points*) I is the only positive unitary.

Problem 4.4. (*4 points*) The map $T\colon M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $A \mapsto A + A^T$ is diagonalizable.

5 Problems

For these problems, H will denote a Hilbert Space.

Problem 5.1. (*2 points*) Let $P, Q \in L(H)$ be projections. Show $P \perp Q$ $(\text{im}(P) \perp \text{im}(Q))$ if and only if $PQ = 0$.

Problem 5.2 (Spectral Mapping Theorem)**.** (*2 points*) Suppose $T \in L(H)$ is unitarily diagonalizable and $f \in F(\text{sp}(T), \mathbb{F})$. Show $\text{sp}(f(T)) = f(\text{sp}(T))$.

Problem 5.3. (*2 points*) For which values of $\theta \in [0, 2\pi)$ is $A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in M_2(\mathbb{R})$ unitarily diagonalizable?

Problem 5.4 (Systems of Matrix Units 2)**.** (*4 points*) For which m do there exist systems of matrix units for $M_n(\mathbb{F}) = L(\mathbb{F}^n)$ of size m? For simplicity, we will identify $A \in M_n(\mathbb{F})$ with $L_A \in L(\mathbb{F}^n)$.