Infinite index subfactors and the GICAR algebras

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Abstract

Given a II_1-subfactor $A \subset B$ of arbitrary index, we show that the GICAR category faithfully embeds as $A-A$ bimodule maps among the bimodules $\bigotimes_A^n L^2(B)$. As a corollary, there is a copy of the GICAR algebra $\bigoplus_{k=0}^n M_{\binom{n}{k}}(\mathbb{C})$ inside $A_0' \cap A_{2n}$. **TODO: what is A_i?**

Contents

1 Introduction

TODO:

Acknowledgements:

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2 Background

In this section, we give the necessary background from [EN96, Bur03]. We will prove one proposition in this section.

Notation 2.1. Let $A_0 = A \subset B = A_1$ be a II_1-subfactor of arbitrary index.

Definition 2.2 ([EN96]).] For $n \in \mathbb{N}$, we inductively define:

- n.f.s. operator valued weights: $T_1 = E_A \colon B \to A$ is the unique trace-preserving conditional expectation, and $T_{n+1} \colon A_{n+1}^+ \to \widehat{A_n^+}$ is the unique trace-preserving operator valued weight, which satisfies

$$T_{n+1}(\Lambda(x)\Lambda(y)^*) = xy^* \text{ for all } x, y \in \mathfrak{n}_{T_{n+1}}$$

- The operators $\Lambda(x) \colon L^2(A_{n-1}, \mathrm{Tr}_{n-1}) \to L^2(A_n, \mathrm{Tr}_n)$ for $x \in \mathfrak{n}_{T_n}$ commute with the right A_{n-1} action, and satisfy
  (i) $\Lambda(x)\widehat{y} = \widehat{xy}$ for all $x \in \mathfrak{n}_{T_n}$ and $y \in \mathfrak{n}_{\mathrm{Tr}_{n-1}}$ (note $xy \in \mathfrak{n}_{T_n} \cap \mathfrak{n}_{\mathrm{Tr}_n}$),
  (ii) $\Lambda(x)^*\widehat{y} = \widehat{T_n(x^*y)}$ for all $x \in \mathfrak{n}_{T_n}$ and $y \in \mathfrak{n}_{\mathrm{Tr}_n}$,
  (iii) $\Lambda(x)^*\Lambda(y) = T(x^*y)$ for all $x, y \in \mathfrak{n}_{T_n}$,
- n.f.s. traces $\mathrm{Tr}_i = \mathrm{tr}_i$ for $i = 0, 1$, and

$$\mathrm{Tr}_{n+1}(L(\xi)L(\xi)^*) = \mathrm{Tr}_{n-1}(L(\xi)^*L(\xi)) = \|\xi\|^2 \text{ for all } \xi \in D(L^2(A_n))_{A_{n-1}}, \text{ or}$$
$$\mathrm{Tr}_{n+1}(\Lambda(x)\Lambda(y)^*) = \mathrm{Tr}_{n-1}(xy^*) \text{ for all } x, y \in \mathfrak{n}_{T_n},$$

- J_n is the conjugate-linear unitary which is the extension of the adjoint on $\mathfrak{n}_{\mathrm{Tr}_n}$,
- The Jones tower

$$\begin{aligned} A_{n+1} = J_n(A'_{n-1} \cap B(L^2(A_n, \mathrm{Tr}_n)))J_n &= \left\{ L(\eta)L(\xi)^* \middle| \eta, \xi \in D(L^2(A_n))_{A_{n-1}} \right\}'' \\ &= \left\{ \Lambda(x)\Lambda(y)^* \middle| x, y \in \mathfrak{n}_{T_{n+1}} \right\}''. \end{aligned}$$

Notation 2.3. We use the following notation:

- $B^n = \bigotimes_A^n \widehat{B} \subset \bigotimes_A^n L^2(B) \cong L^2(A_n)$ via isomorphisms θ_n (see [Bur03]).
- For $x_1, \ldots x_n \in B$, we write $\widehat{x_1} \otimes \cdots \otimes \widehat{x_n} \in B^n$ omitting the subscript A to distinguish between operators and vectors, such as $x \otimes_A \mathrm{id}_1$ and $\widehat{x} \otimes \widehat{1}$ for $x \in B$. One is left multiplication by $x \in B$ on $L^2(B) \otimes_A L^2(B)$, and the other is $\theta_2^{-1}(\widehat{xe_1})$.

Facts 2.4.

- *[Multistep basic construction, [EN96]] For $0 \leq k \leq n$, the inclusions $A_{n-k} \subseteq A_n \subseteq A_{n+k}$ are standard, i.e.*

$$(\mathrm{id}_k \otimes_A J_{n-k}A_{n-k}J_{n-k})' = J_n(A_{n-k} \otimes_A \mathrm{id}_k)'J_n \cong A_{n+k}.$$

- *[Shifts, [EN96]] For $0 \leq k \leq n$, we let $j_n = (x \mapsto J_n x^* J_n)$ for $x \in B(L^2(A_n, \mathrm{Tr}_n))$. Then j_n is an anti-isomorphism of A_{n-k} onto A'_{n+k}. Hence if we compose two j_ℓ's, we get an isomorphism:*

$$j_{n+1}j_n(A'_{n-k} \cap A_n) \underset{anti}{\cong} j_{n+1}(A'_n \cap A_{n+k}) \underset{anti}{\cong} A'_{n-k+2} \cap A_{n+2}.$$

- *[Odd Jones projections, [Bur03]] For all $n \in \mathbb{N}$, $\mathrm{Tr}_{2n}|_{A^+_{2n-1}} = \infty$ and $\mathrm{Tr}_{2n+1}|_{A^+_{2n}} = \mathrm{Tr}_{2n}$. Therefore, $T_{2n+1}\colon A_{2n+1} \to A_{2n}$ is a conditional expectation, which gives rise to the odd Jones projection e_{2n+1}.*

  When we realize A_{2n+1} acting on $L^2(A_{n+1}, \mathrm{Tr}_{n+1})$ from the multistep basic construction, $e_{2n+1} = J_{n+1} e_1 J_{n+1}$.

Proposition 2.5. *When we realize A_{2n} acting on $\bigotimes_A^n L^2(B)$, for $1 \leq i \leq n$, we may identify the Jones projection e_{2i-1} with $\mathrm{id}_{i-1} \otimes_A e_1 \otimes_A \mathrm{id}_{n-i}$*

Proof. We use strong induction on n. The case $n = 1$ is trivial. Suppose the result holds for all $0 \leq i < n$. We now consider A_{2n+2} acting on $\bigotimes_A^{n+1} L^2(B)$. Note that $A_{2n} \hookrightarrow A_{2n+2}$ by $x \mapsto x \otimes_A \mathrm{id}_1$. Hence the result is true for all $1 \leq i \leq n$. To show the result is true for $i = n+1$, we simply note by the isomorphism $L^2(A_{n+1}, \mathrm{Tr}_{n+1}) \cong \bigotimes_A^{n+1} L^2(B)$,

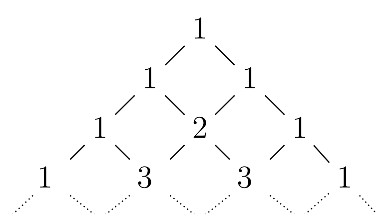
$$e_{2n+1} = J_{n+1}(e_1 \otimes_A \mathrm{id}_n) J_{n+1} = \mathrm{id}_n \otimes_A e_1.$$

□

3 The GICAR categories

In this subsection, we give a diagrammatic representation of the GICAR (Gauge-invariant canonical anti-commutation relations) categories. We refer the reader to Davidson [Dav96] for a general discussion of the AFD GICAR C^*-algebra. We will only need the sequence of finite dimensional algebras approximating it.

Definition 3.1. The tower of complex GICAR algebras $(G_n)_{n \geq 0}$ is given by the Bratteli diagram corresponding to Pascal's Triangle:



We do not specify a sequence of traces on the tower.

Notation 3.2. Given a category $\mathcal{C}$, we write $X, Y \in \mathcal{C}$ to denote X, Y are objects in $\mathcal{C}$, and we write $\mathcal{C}(X, Y)$ for the space of morphisms from X to Y. If the objects in $\mathcal{C}$ are symbols of the form $[n]$ for $n \geq 0$, we simply write $\mathcal{C}(m, n)$ for $\mathcal{C}([m], [n])$ as in [Pen11].

We will now define two small categories, the diagrammatic GICAR category $\mathcal{G}^d$ and the abstract GICAR category $\mathcal{G}$, and show they are equivalent.

Definition 3.3. Let the diagrammatic GICAR category $\mathcal{G}^d$ be the following small involutive tensor category:

Objects: $[n]$ for $n \geq 0$, and

Morphisms: The morphisms from $[m]$ to $[n]$ are the $\mathbb{C}$-linear combinations on isotopy classes of tangles with

- no input disks, and one external disk, drawn in the shape of a rectangle. It has m lower boundary points, and n upper boundary points,
- each boundary point is connected to one string. Strings cannot intersect. There are three possibilities for strings:

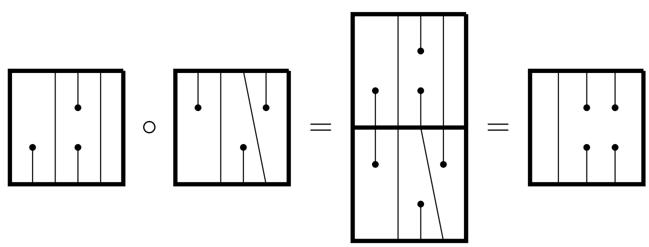
  (1) the string connects to a lower boundary point and an upper boundary point and is called a through string,

  (2) the string connects to only a lower boundary point and is called a cap, or

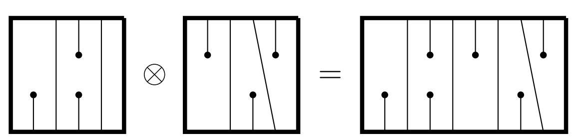
  (3) the string connects to only an upper boundary point and is called a cup.

  We draw a dark circle on the end of the string to denote that that end does not attach to another boundary point.

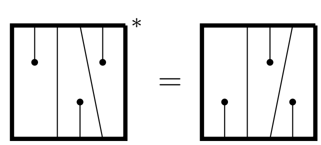
Composition: Composition is the $\mathbb{C}$-linear extension vertical stacking. When we get a floating string (a string connected to no boundary points), we just remove it.



Tensor: $[m] \otimes [n] = [m + n]$ and morphisms are tensored by the $\mathbb{C}$-linear extension of horizontal join.

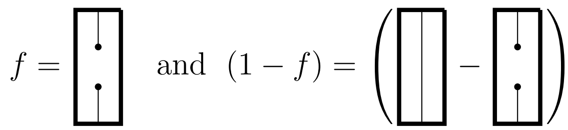


Adjoint: The adjoint map is the $\mathbb{C}$-linear extension of reflection about the x-axis.

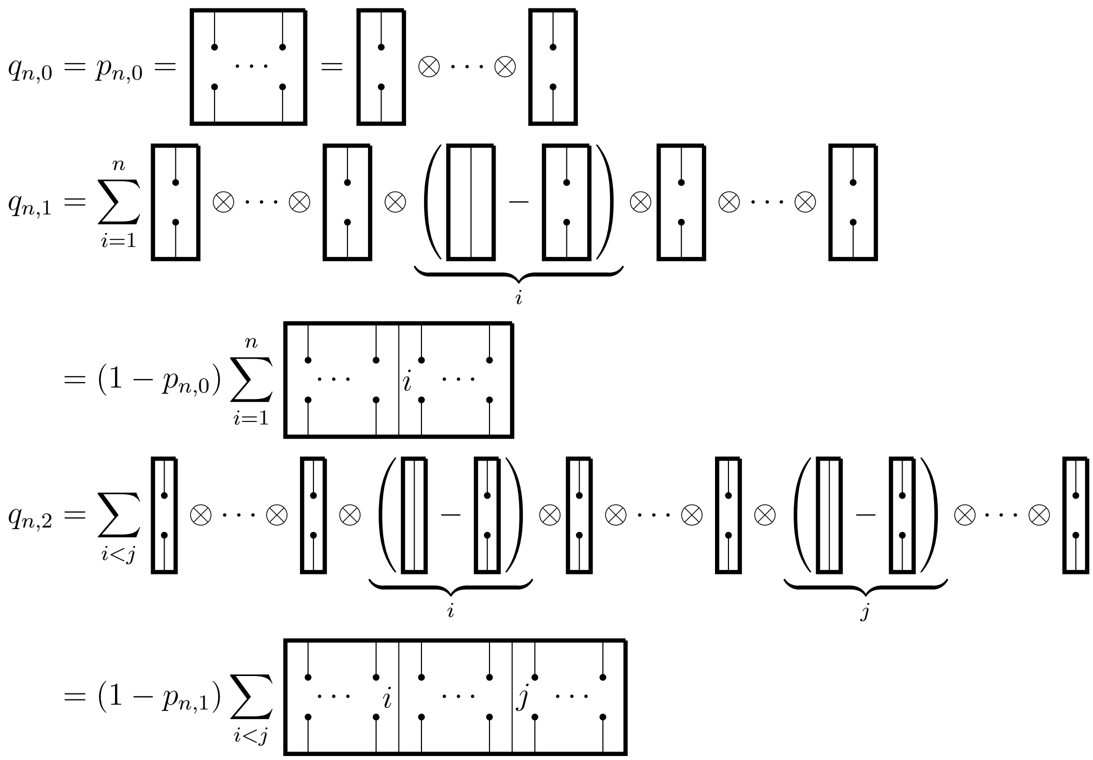


Remark 3.4. Let $0 \leq k \leq n$. There are exactly $\binom{n}{k}$ diagrams with exactly k through strings in $\mathcal{G}^d(n,n)$. Hence $\dim_{\mathbb{C}}(\mathcal{G}(n,n)) = \sum_{k=0}^{n} \binom{n}{k}$. Moreover, the diagrams with exactly k through strings in $\mathcal{G}^d(n,n)$ span a full matrix algebra of dimension $\binom{n}{k}$.

Let $\mathcal{G}_{k,n} \subset \mathcal{G}(n,n)$ denote the complex span of the diagrams with less than or equal to k through strings. Note that $\mathcal{G}_{n,k}\mathcal{G}_{n,m} \subseteq \mathcal{G}_{n,\min\{k,m\}}$. Let $p_{n,k}$ be the projection onto $\mathcal{G}_{n,k}$, and let $q_{n,k} = p_{n,k} - p_{n,k+1}$, with the convention that $p_{n,-1} = p_{n,n+1} = 0$. It is clear that n-fold simple tensors composed of

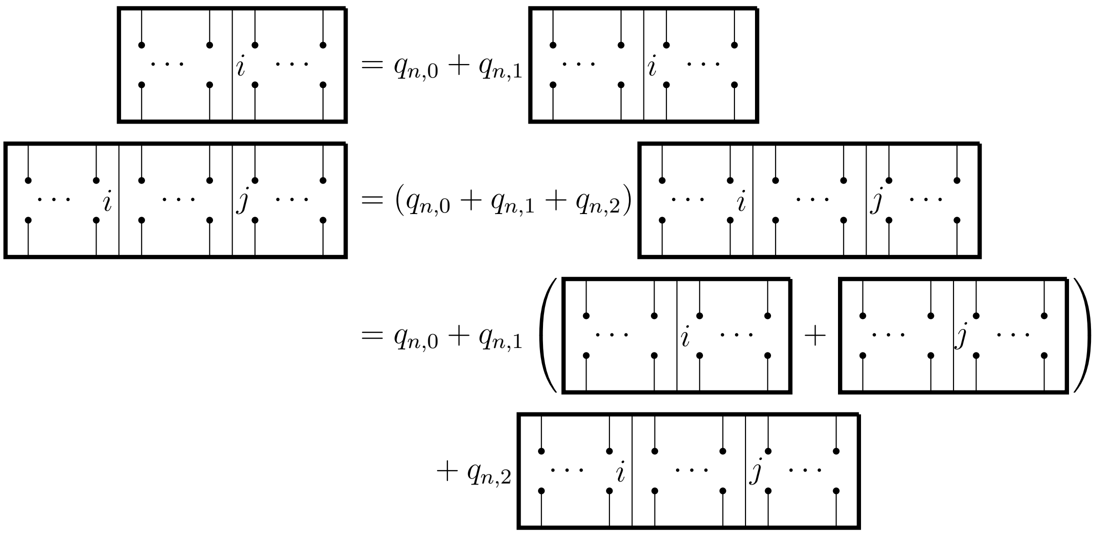


are nonzero pairwise orthogonal projections in $\mathcal{G}(n, n)$, and there are exactly $\sum_{k=0}^{n} \binom{n}{k}$ of them (for $k = 0, \ldots, n$, we choose k positions for $(1 - f)$ in the simple tensor). By counting dimensions, these must be all the minimal projections of $\mathcal{G}(n, n)$. By induction on k, we have that all such simple tensors where we choose k positions for $(1 - f)$ are the minimal projections under $q_{n,k}$. This implies



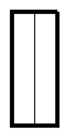
and so forth. By induction on k, $(1 - p_{n,k})\mathcal{G}_{n,k} = q_{n,k}\mathcal{G}_{n,k} \cong M_{\binom{n}{k}}(\mathbb{C})$, which is orthogonal to $\mathcal{G}_{n,k-1}$. Hence $\mathcal{G}^d(n, n) \cong \bigoplus_{k=0}^{n} M_{\binom{n}{k}}(\mathbb{C})$.

Suppose now that $d \in \mathcal{G}^d(n, n)$ is a string diagram with m through strings. Then if $m < k$, $q_{n,k}d = 0$. If $m = k$, then $q_{n,k}d$ is the simple tensor of f's and $(1 - f)$'s where we replace d's through strings with $(1 - f)$'s. If $m > k$, then $q_{n,k}d$ is the sum of all ways to replace k of d's through strings with $(1 - f)$'s and replace the rest of the through strings with f's. Hence

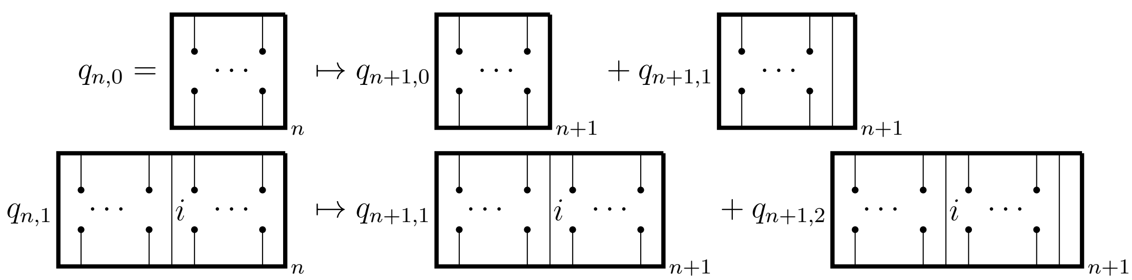


and so forth.

Now we include $\mathcal{G}^d(n,n) \to \mathcal{G}^d(n+1,n+1)$ (as $*$-algebras) by adding a through string on the right

$x \mapsto x \otimes$  ,

and we get inclusions $\mathcal{G}_{n,k} \to \mathcal{G}_{n+1,k+1}$. It is clear that a simple tensor of f's and $(1-f)$'s under $q_{n,k}$ maps to a sum of two simple tensors of f's and $(1-f)$'s, one under $q_{n+1,k}$ and one under $q_{n+1,k+1}$. Using diagrams (where the subscript denotes half the number of total boundary points),



and so forth, and we see that the Bratteli diagram for the tower $(\mathcal{G}^d(n,n))_{n\geq 0}$ with this inclusion is Pascal's Triangle as in Definition 3.1. The same argument also holds for the opposite inclusion by adding a through string on the left.

We have just proven the following:

Theorem 3.5. *There are $*$-algebra isomorphisms $\mathcal{G}^d(n,n) \to G_n$ which preserve the inclusion. The same result is true (for different isomorphisms) when we include by adding a string on the left.*

Definition 3.6. Let the abstract GICAR category $\mathcal{G}$ be the following small involutive tensor category:

Objects: symbols $[n_1] \otimes \cdots \otimes [n_k]$ for $n_i \geq 0$ for all $i = 1, \ldots, k$ where $k \geq 1$.

Tensoring objects: For all $n_1, n_2 \geq 0$, there are fixed isomorphisms $[n_1] \otimes [n_2] \cong [n_1 + n_2]$ which satisfy the pentagon and triangle idenitities (which gives associator isomorphisms).

Morphisms: generated by $\mathbb{C}$-linear combinations of tensors of identity maps, associators, and the morphisms $a\colon [1] \to [0]$ and $a^*\colon [0] \to [1]$ subject to the relation $aa^* = \mathrm{id}_{[0]}$.

Adjoint: The adjoint map fixes all objects, and on morphisms, it is the anti-linear extension of $a \mapsto a^*, a^* \mapsto a$, $(\varphi \otimes \psi)^* = \varphi^* \otimes \psi^*$ for morphisms φ, ψ, and if φ, ψ are composable, $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$.

Definition 3.7. For $n \geq 0$ and $i = 1, \ldots, n$, we define $a_i\colon [n] \to [n-1]$ by the following commutative diagram:

$$\begin{array}{ccc} [n] & \xrightarrow{\cong} & [i-1] \otimes [1] \otimes [n-i] \\ {\scriptstyle a_i}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{[i-1]} \otimes a \otimes \mathrm{id}_{[n-i]}} \\ [n-1] & \xrightarrow{\cong} & [i-1] \otimes [0] \otimes [n-i] \end{array}$$

where we do not specify the associator isomorphisms.

Compare the following proposition with Proposition 4.6.

Proposition 3.8.

(1) The a_i, a_j^* satisfy the following relations:

(i) $a_i a_j = a_{j-1} a_i$ and $a_i^* a_j^* = a_j^* a_{i-1}^*$ for all $i < j$,

(ii) $a_j a_i^* = \begin{cases} a_i^* a_{j+1} & \text{if } i < j \\ \mathrm{id}_{[n]} & \text{if } i = j \\ a_{i+1}^* a_j & \text{if } i > j \end{cases}$

(2) The words on the a_i, a_j^* give bases for $\mathcal{G}(m, n)$.

(3) As in [Pen11], any word in the a_i, a_j^* has a unique standard representation of the form

$$a_{i_k}^* \cdots a_{i_1}^* a_{j_1} \cdots a_{j_\ell}$$

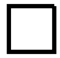
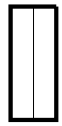
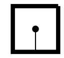
where $i_k > i_{k-1} > \cdots > i_1$ and $j_1 < j_2 < \cdots j_\ell$. In particular, $\dim(\mathcal{G}(m, n)) < \infty$ for all m, n.

Proof. Straightforward. □

Theorem 3.9. *The involutive categories $\mathcal{G}$ and $\mathcal{G}^d$ are equivalent via the $*, \otimes$-functor $\Psi\colon \mathcal{G} \to \mathcal{G}^d$ given by the $\mathbb{C}$-linear $*, \otimes$-extension of*

Objects: $\Psi([n_1] \otimes \cdots \otimes [n_k]) = [n_1 + \cdots + n_k]$

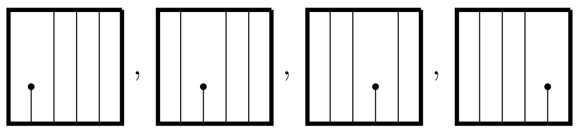
Morphisms: Ψ *of a morphism is determined by*

$\Psi(\mathrm{id}_{[0]}) =$ , $\Psi(id_{[1]}) =$  *and* $\Psi(a) =$ .

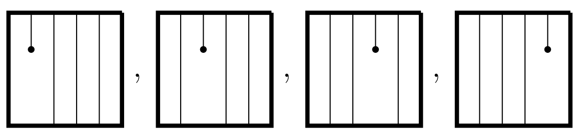
Hence we may confuse $\mathcal{G}$ with $\mathcal{G}^d$ in the sequel.

Proof. Clearly Ψ is a $*, \otimes$-functor. To construct a functor Ψ^{-1} such that $\Psi^{-1} \circ \Psi \cong \mathrm{id}_{\mathcal{G}}$ and $\Psi \circ \Psi^{-1} = \mathrm{id}_{\mathcal{G}^d}$, we use techniques of [Pen11]. Note that

- $a_i \in \mathcal{G}(n, n-1)$ maps to the diagram with n lower boundary points, $n-1$ upper boundary points, a cap attached to lower boundary point i, and all other boundary points connected by through strings.

$\mathcal{G}(4,3) \ni a_1^*, \ldots, a_4^* \longmapsto$  $\in \mathcal{G}^d(4,3)$

- $a_j \in \mathcal{G}(n, n+1)$ maps to the diagram with n lower boundary points, $n+1$ upper boundary points, a cup attached to upper boundary point j, and all other boundary points connected by through strings.

$\mathcal{G}(3,4) \ni a_1, \ldots, a_4 \longmapsto$  $\in \mathcal{G}^d(3,4)$

We define the inverse Ψ^{-1} by the above. Given a string diagram d, $\Psi^{-1}(d)$ is the word on a_i, a_j^* in standard form

$$a_{i_k}^* \cdots a_{i_1}^* a_{j_1} \cdots a_{j_\ell}$$

where $i_1 < \cdots < i_k$ are the positions of the cups of d and $j_1 < \cdots < j_\ell$ are the positions of the caps of d. That $\Psi^{-1} \circ \Psi \cong \mathrm{id}_{\mathcal{G}}$ and $\Psi \circ \Psi^{-1} = \mathrm{id}_{\mathcal{G}^d}$ follows immediately from Proposition 3.8. □

4 The GICAR category and the tower $(A_0' \cap A_{2n})_{n \geq 0}$

In this subsection, we show the GICAR category $\mathcal{G}$ is faithfully represented as $A-A$ bimodule maps on the even half of the Jones tower $(A_0' \cap A_{2n})_{n\geq 0}$. In particular, we will have an injection $G_n \hookrightarrow A_0' \cap A_{2n}$.

Notation 4.1. We write $e_A \colon L^2(B) \to L^2(A)$ for the canonical projection, so $e_A^* \colon L^2(A) \to L^2(B)$ is the inclusion. Note $e_1 = e_A^* e_A \in B(L^2(B))$.

Definition 4.2. Let $\mathcal{G}^{A \subset B}$ be the following small involutive tensor categroy:

Objects: all finite $\otimes_A$'s of $L^2(A)$ and $L^2(B)$. There are natural isomorphisms between the relative tensor powers of $L^2(A), L^2(B)$.

Tensoring objects: Given by $\otimes_A$.

Morphisms: $\mathbb{C}$-linear combinations of basis elements generated by tensors of identity morphisms, associators, $e_A\colon L^2(B) \to L^2(A)$, and $e_A^*\colon L^2(A) \to L^2(B)$ where the tensoring of morphisms is $\otimes_A$.

Composition: The composition is the composition of operators.

Adjoint: The adjoint is the adjoint of operators.

Remark 4.3. Note that all morphisms in $\mathcal{G}^{A\subset B}$ are $A-A$ bimodule maps.

Definition 4.4. For $n \geq 0$ and $i = 1, \ldots, n$, define the maps $\mathfrak{a}_i\colon \bigotimes_A^n L^2(B) \to \bigotimes_A^{n-1} L^2(B)$ by the following commutative diagrams:

$$\begin{array}{ccc} \bigotimes_A^n L^2(B) & \xrightarrow{\cong} & \left(\bigotimes_A^{i-1} L^2(B)\right) \otimes_A L^2(B) \otimes_A \left(\bigotimes_A^{n-i} L^2(B)\right) \\ \downarrow{\scriptstyle \mathfrak{a}_i} & & \downarrow{\scriptstyle \mathrm{id}_{i-1}\otimes_A e_A \otimes_A \mathrm{id}_{n-i}} \\ \bigotimes_A^{n-1} L^2(B) & \xrightarrow{\cong} & \left(\bigotimes_A^{i-1} L^2(B)\right) \otimes_A L^2(A) \otimes_A \left(\bigotimes_A^{n-i} L^2(B)\right). \end{array}$$

As in Definition 3.7, we do not specify the associator isomorphisms.

The following two propositions are straightforward calculations. Compare Propositions 3.8 and 4.6.

Proposition 4.5. *The maps $\mathfrak{a}_i$, $\mathfrak{a}_i^*$ are given by the unique extensions of*

$$\mathfrak{a}_i(\widehat{x_1} \otimes \cdots \otimes \widehat{x_n}) = \widehat{x_1} \otimes \cdots \otimes \widehat{x_{i-1}} \otimes \widehat{E_A(x_i)x_{i+1}} \otimes \cdots \otimes \widehat{x_n} \qquad \text{(annihilation)}$$
$$\mathfrak{a}_i^*(\widehat{x_1} \otimes \cdots \otimes \widehat{x_n}) = \widehat{x_1} \otimes \cdots \otimes \widehat{x_{i-1}} \otimes \widehat{1} \otimes \widehat{x_i} \otimes \cdots \otimes \widehat{x_n} \qquad \text{(creation)}$$

where $x_i \in B$ for all i.

Proposition 4.6.

(1) The words on $\mathfrak{a}_i, \mathfrak{a}_j^*$ satisfy the following relations:

(i) $\mathfrak{a}_i^*\mathfrak{a}_j^* = \mathfrak{a}_j^*\mathfrak{a}_{i-1}^*$ and $\mathfrak{a}_i\mathfrak{a}_j = \mathfrak{a}_{j-1}\mathfrak{a}_i$ for all $i < j$,

(ii) $\mathfrak{a}_j\mathfrak{a}_i^* = \begin{cases} \mathfrak{a}_i^*\mathfrak{a}_{j+1} & \text{if } i < j \\ 1 & \text{if } i = j \\ \mathfrak{a}_{i+1}^*\mathfrak{a}_j & \text{if } i > j \end{cases}$ *and*

(iii) $\mathfrak{a}_i^*\mathfrak{a}_i = \mathrm{id}_{i-1} \otimes_A e_1 \otimes_A \mathrm{id}_{n-i}$ for all $i \leq n$.

(2) The words on the $\mathfrak{a}_i, \mathfrak{a}_j^*$ span $\mathcal{G}^{A\subset B}(m,n)$.

(3) *Each word in the* $\mathfrak{a}_i, \mathfrak{a}_j^*$ *has a unique standard form*

$$\mathfrak{a}_{i_k}^* \cdots \mathfrak{a}_{i_1}^* \mathfrak{a}_{j_1} \cdots \mathfrak{a}_{j_\ell}$$

where $i_k > \cdots > i_1$ *and* $j_1 < \cdots < j_\ell$.

Proof. Straightforward from Proposition 4.5. □

Remark 4.7. Recall that $A_0' \cap A_{2n} \cong \mathrm{End}_{A-A}(L^2(A_n)) \cong \mathrm{End}_{A-A}(\bigotimes_A^n L^2(B))$. Hence for $1 \leq i, j \leq n$, $\mathfrak{a}_i^* \mathfrak{a}_j \in A_0' \cap A_{2n}$.

Corollary 4.8. *For all* $1 \leq i, j \leq n$, *the maps* $\mathfrak{a}_i \mathfrak{a}_j^*, \mathfrak{a}_j^* \mathfrak{a}_i \in A_0' \cap A_{2n}$ *witness the von Neumann equivalence of the projections* e_{2i-1}, e_{2j-1} *in* $A_0' \cap A_{2n}$. *Hence* $A_0' \cap A_{2n}$ *is not abelian for* $n \geq 2$.

Proof. By Propositions 2.5 and 4.6,

$$(\mathfrak{a}_i^* \mathfrak{a}_j)(\mathfrak{a}_j^* \mathfrak{a}_i) = \mathfrak{a}_i^* \mathfrak{a}_i = e_{i-1} \otimes_A e_1 \otimes_A \mathrm{id}_{n-i} = e_{2i-1} \text{ and}$$
$$(\mathfrak{a}_j^* \mathfrak{a}_i)(\mathfrak{a}_i^* \mathfrak{a}_j) = \mathfrak{a}_j^* \mathfrak{a}_j = e_{j-1} \otimes_A e_1 \otimes_A \mathrm{id}_{n-j} = e_{2j-1}.$$

□

Lemma 4.9. *Suppose*

$$x = \mathfrak{a}_{i_k}^* \cdots \mathfrak{a}_{i_1}^* \mathfrak{a}_{j_1} \cdots \mathfrak{a}_{j_\ell} \in \mathcal{G}^{A \subset B}(n, n-\ell+k)$$

is in the standard form of Proposition 4.6. Then there are $\xi \in B^n$ *and* $\eta \in B^{n-\ell+k}$ *such that* $\langle x\xi, \eta \rangle = 1$ *and* $\langle y\xi, \eta \rangle = 0$ *for all words* $y \in \mathcal{G}^{A \subset B}(n, n-\ell+k)$ *on the* $\mathfrak{a}_i, \mathfrak{a}_j^*$ *whose standard form has length at least* $\ell + k$.

Proof. Suppose $\{b\}$ is an orthonormal basis for B over A containing 1, which is always possible (see Proposition 3.2.20 of [Bur03]). Pick $b \neq 1$ from the basis. Let

- $\xi \in B^n$ be the simple tensor with $\widehat{1}$'s in positions $j_1 < \cdots < j_\ell$ and $\widehat{b}$'s otherwise, and
- $\eta \in B^{n-\ell+k}$ be the simple tensor with $\widehat{1}$'s in positions $i_1 < \cdots < i_k$ and $\widehat{b}$'s otherwise.

Then $\langle x\xi, \eta \rangle = 1$. Suppose $y \in \mathcal{G}^{A \subset B}(n, n-\ell+k)$ is a word on the $\mathfrak{a}_i, \mathfrak{a}_j^*$ with $\langle y\xi, \eta \rangle \neq 0$, and write w' in standard form

$$y = \mathfrak{a}_{i'_{k'}}^* \cdots \mathfrak{a}_{i'_1}^* \mathfrak{a}_{j'_1} \cdots \mathfrak{a}_{j'_{\ell'}}.$$

Since $E_A(b) = 0$, we must have $j'_1, \cdots j'_{\ell'} \in \{j_1, \ldots, j_\ell\}$ and $i'_1, \cdots i'_{k'} \in \{i_1, \ldots, i_k\}$, so $k' \leq k$ and $\ell' \leq \ell$. Moreover, if $\ell' = \ell$ and $k' = k$, then $y = x$. □

Corollary 4.10. *The words on* $\mathfrak{a}_i, \mathfrak{a}_j^*$ *in* $\mathcal{G}^{A \subset B}(m, n)$ *are linearly independent.*

Proof. Suppose $0 = \sum_{i=1}^{k} \lambda_i w_i \in \mathcal{G}^{A\subset B}(m,n)$ where $w_i \in \mathcal{G}^{A\subset B}(m,n)$ are distinct words on the $\mathfrak{a}_i, \mathfrak{a}_j^*$. We may assume that the words are ordered by increasing standard form word length. We show by induction on k that all the λ_i's are zero. If $k = 1$, this is trivial, since $w \neq 0$ for all words w by Lemma 4.9 (there is a linear functional which separates w from 0). Suppose now that $k > 1$. Since the standard form word length of w_1 is minimal, by Lemma 4.9, there are $\xi \in B^m$ and $\eta \in B^n$ such that $\langle w_i\xi, \eta\rangle = \delta_{1,i}$. This means

$$\lambda_1 = \sum_{i=1}^{k} \lambda_i \langle w_i\xi, \eta\rangle = \left\langle \sum_{i=1}^{k} \lambda_i w_i \xi, \eta \right\rangle = 0.$$

We are finished by the induction hypothesis. □

Theorem 4.11. *The $*, \otimes$-functor $\Phi\colon \mathcal{G} \to \mathcal{G}^{A\subset B}$ given by $[n] \mapsto \bigotimes_A^n L^2(B)$ for $n \geq 0$, $a \mapsto e_A$ defines an equivalence of involutive tensor categories.*

Moreover, if q is a minimal projection under $q_{n,k} \in \mathcal{G}(n,n)$, then

(1) $\Phi(q)$ is a simple tensor in $\bigotimes_A^n Q_1 \subset Q_n$ composed of k copies of $1 - e_1$ and $n-k$ copies of e_1, and

(2) $\Phi(q_{n,k})$ is the sum of all such simple tensors.

Proof. By Proposition 4.6, the relations of $\mathcal{G}$ are satisfied in $\Phi(\mathcal{G})$, so Φ is well-defined and preserves the adjoint and the tensor structure. Hence (1) and (2) are immediate from Propostion 4.6 and Remark 3.4. Φ is clearly essentially surjective, and Φ is fully faithful (injective on hom spaces) by Corollary 4.10. □

Corollary 4.12. *The involutive tensor category $\mathcal{G} \cong \mathcal{G}^d$ is positive, i.e., if $x \in \mathcal{G}(m,n)$ and $x^*x = 0 \in \mathcal{G}(m,m)$, then $x = 0$.*

Proof. If $x \in \mathcal{G}(m,n)$ with $x^*x = 0$, then $\Phi(x^*x) = 0$, so $\Phi(x) = 0$ as $\mathcal{G}^{A\subset B}$ is positive. Hence $x = 0$ as Φ is injective on hom spaces. □

Corollary 4.13. *Suppose $0 \leq k \leq n$ and $q \leq q_{n,k}$ is a minimal projection. Then*

$$\operatorname{Tr}_n(\Phi(q)) = \begin{cases} \operatorname{Tr}_n(e_1 \otimes_A \cdots \otimes_A e_1) = 1 & \text{if } q = q_{n,0} \\ \infty & \text{else.} \end{cases}$$

The same holds for $\operatorname{Tr}_n^{\mathrm{op}}$.

Proof. Follows easily from Theorem 4.11 using the planar calculus of []. **TODO:** □

Remark 4.14. The results of this section also show that if $A \subset B$ is a finite index II_1-subfactor with $[B\colon A] > 1$, then $\mathcal{G}$ is equivalent to $\mathcal{G}^{A\subset B}$. In fact, one can show $\Phi(\mathcal{G}(n,n)) \cong G_n$ by noting that the minimal projections on the two sides of the Pascal's Triangle correspond to $e_1e_3\cdots e_{2n+1}$ and $e_1^\perp e_3^\perp \cdots e_{2n+1}^\perp$. An easy exercise using the Markov trace on the centralizer algebras $A_0' \cap A_{2n}$ shows that

$$\operatorname{tr}_n({}_1e_3\cdots e_{2n+1}) = \tau^n \text{ and } \operatorname{tr}_n(e_1^\perp e_3^\perp \cdots e_{2n+1}^\perp) = (1-\tau)^n$$

where $\tau = [B\colon A]^{-1}$. Since $\tau^n \neq 0 \neq (1-\tau)^n$, $\Phi(\mathcal{G}(n,n)) \cong G_n$.

TODO: Note on unitary 2-categories. In finite index, get unitarity and duals. For infinite index, don't get evaluation and coevaluation for $L^2(B) \subset L^2\langle B, e_A\rangle$

References

[Bur03] Michael Burns, *Subfactors, planar algebras, and rotations*, 2003, Ph.D. Thesis at the University of California Berkeley.

[Dav96] Kenneth R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996, MR1402012. MR 1402012 (97i:46095)

[EN96] Michel Enock and Ryszard Nest, *Irreducible inclusions of factors, multiplicative unitaries, and Kac algebras*, J. Funct. Anal. **137** (1996), no. 2, 466–543, MR1387518.

[Pen11] David Penneys, *A cyclic approach to the annular Temperley-Lieb category*, J. Knot Theory and its Ramifications (2011), arXiv:0912.1320, DOI:10.1142/S0218216511010012.