1 Sharp A_2 inequality for Haar shift operators

after M. Lacey, S. Petermichl and M. C. Reguera [2]
A summary written by Diogo Oliveira e Silva

Abstract

The authors of [2] prove linear growth in the A_2 characteristic for weighted L^2 inequalities involving Haar shift operators. We describe how two new ingredients of the proof, a two weight $T1$ theorem and a corona decomposition of the weight, come into play in the proof.

1.1 Introduction

It is a classical result [5] that the Hilbert transform H is bounded on $L^p(w)$ if and only if $w \in A_p$. Specializing to $p = 2$, the weight w satisfies

$$[w]_{A_2} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w\Big)\Big(\frac{1}{|Q|}\int_Q w^{-1}\Big) < \infty$$

if and only if we have an inequality

$$\|Hf\|_{L^2(w)} \leq C\|f\|_{L^2(w)},$$

where the constant C depends on the A_2-constant $[w]_{A_2}$. Determining the exact dependent of C on $[w]_{A_2}$ is a difficult problem with nontrivial consequences. In [4] the following optimal bound is proved:

$$\|Hf\|_{L^2(w)} \lesssim [w]_{A_2}\|f\|_{L^2(w)}. \tag{1}$$

Later work showed that if one replaces H by a Riesz transform R_j or the Beurling-Ahlfors transform B, inequality (1) still holds. For applications of these results to the theory of elliptic PDE, see [4].

Earlier proofs made use of Haar shift operators (which are entirely natural in this context since H, R_j and B are all obtained by appropriate averaging of Haar shifts) together with Bellman function techniques. The present paper [2] still deals with Haar shift operators, but instead of Bellman functions uses a deep two weight $T1$ theorem from [3], together with an appropriate corona decomposition of the weight w to verify the relevant Carleson measure estimates.

1.2 Haar shift operators

We will denote the family of all dyadic cubes in $\mathbb{R}^d$ by $\mathcal{Q}$, and all cubes will from now onwards be assumed dyadic. For $Q \in \mathcal{Q}$ and $m \in \mathbb{N}$, we let $Q^{(m)}$ be the m-fold parent of Q. By a Haar function h_Q on a cube Q we mean any function supported on Q which is constant on dyadic subcubes of Q and which satisfies the following conditions:

(i) $\int_Q h_Q = 0$; (cancellation)

(ii) $\|h_Q\|_\infty \leq |Q|^{-1/2}$. (size)

In particular, Haar functions are L^2 normalized. They play an essential role in the following definition:

Definition 1. *We say that T is a Haar shift operator (HSO) of index $\tau \in \mathbb{N}_0$ on $\mathbb{R}^d$ if $Tf = \sum_{Q\in\mathcal{Q}} \langle f, g_Q\rangle \gamma_Q$, where*

$$g_Q, \gamma_Q \in \mathit{span}\{h_{Q'} : Q' \subset Q, 2^{-\tau d}|Q| \leq |Q'|\}, \text{ and} \tag{2}$$

$$\|g_Q\|_\infty, \|\gamma_Q\|_\infty \leq |Q|^{-1/2}. \tag{3}$$

Let us turn to some illustrative examples of Haar shift operators.

Example 2. *(HSO of index 0) Given a bounded sequence $\alpha = (\alpha_Q)_{Q\in\mathcal{Q}}$, consider the Haar multiplier T^α given by*

$$T^\alpha f = \sum_{Q\in\mathcal{Q}} \alpha_Q \langle f, h_Q\rangle h_Q.$$

If $\alpha \subset \{-1, 1\}^{\mathbb{N}}$, then T^α is called the Martingale transform, also known as the dyadic Hilbert transform.

Example 3. *(HSO of index 1) For a one-dimensional example, consider the Haar shift S given by*

$$Sf = \sum_{I\in\mathcal{Q}} \langle f, h_I\rangle (h_{I_r} - h_{I_l}),$$

where I_r and I_l denote the right and left halves of the interval $I \subset \mathbb{R}$, respectively.

The purpose of conditions (2) and (3) is to ensure that HSOs are Calderón-Zygmund operators. That is the content of the following Proposition:

Proposition 4. *Let T be a Haar shift operator of index τ on $\mathbb{R}^d$. Then T is bounded on $L^2(dx)$ with norm $\lesssim \tau$, and T maps $L^1(dx)$ into $L^{1,\infty}(dx)$ with norm $\lesssim 2^{\tau d}$.*

We mention some elements that go into the proof of Proposition 4. The cancellation condition (2) allows us to use the lemma of Cotlar, Knapp and Stein, which together with the size condition (3) implies $\|T\|_{L^2(dx)} \lesssim \tau$. The weak (1,1) bound is obtained by the usual Calderón-Zygmund decomposition, where the analysis of the "bad" part is simplified by noting that

$$x \notin Q^{(\tau)} \Rightarrow T(1_Q b)(x) = 0.$$

1.3 Main result and tools

The main result of [2] is the following:

Theorem 5. *Let T be a Haar shift operator of index τ on $\mathbb{R}^d$, and let w be an A_2 weight. Then*

$$\|T\|_{L^2(w)\to L^2(w)} \lesssim_{d,\tau} [w]_{A_2}. \tag{4}$$

We will sketch the proof given in [2] in the next section. For now we explore two essential ingredients which go into the proof of Theorem 5.

1.3.1 A two weight $T1$ theorem

The paper [3] gives an elegant characterization of some two weight inequalities in L^2. Its main result implies the following theorem:

Theorem 6. *Let T be a Haar shift operator of index τ on $\mathbb{R}^d$, and let σ and μ be two positive measures. The inequality*

$$\|T(\sigma f)\|_{L^2(\mu)} \lesssim \|f\|_{L^2(\sigma)}$$

holds if and only if there exist constants $C_1, C_2, C_3 < \infty$ such that for all cubes Q, Q', Q'' with $Q', Q'' \subset Q$ and $2^{-\tau d}|Q| \le |Q'|, |Q''|$,

$$\Big| \int_{Q''} T(\sigma 1_{Q'}) d\mu \Big| \le C_1 \sigma(Q')^{1/2} \mu(Q'')^{1/2}, \tag{5}$$

$$\begin{aligned} \|T(\sigma 1_Q)\|_{L^2(Q,\mu)} &\leq C_2 \sigma(Q)^{1/2} \text{ and} \\ \|T^*(\mu 1_Q)\|_{L^2(Q,\sigma)} &\leq C_3 \mu(Q)^{1/2}. \end{aligned} \tag{6}$$

As a corollary of the proof, one gets that $\|T(\sigma \cdot)\|_{L^2(\sigma)\to L^2(\mu)} \lesssim C_1 + C_2 + C_3$.

1.3.2 The corona decomposition

The following somewhat involved definition will play an essential role in the proof of the main estimate:

Definition 7. *Let $\mathcal{Q}' \subset \mathcal{Q}$ be any bounded collection of cubes, and let μ be a positive measure. Let $\mathcal{L} \subset \mathcal{Q}'$. We call $(\mathcal{Q}'(L))_{L\in\mathcal{L}}$ a corona decomposition of $\mathcal{Q}'$ with respect to μ if the following conditions hold:*

(i) For every $Q \in \mathcal{Q}'$, there exists $L \in \mathcal{L}$ such that $Q \subset L$. Let $\lambda(Q) \in \mathcal{L}$ denote the minimal cube which contains Q, and set $\mathcal{Q}'(L) := \{Q \in \mathcal{Q}' : \lambda(Q) = L\}$. Then

$$\frac{\mu(Q)}{|Q|} \leq 4\frac{\mu(\lambda(Q))}{|\lambda(Q)|};$$

(ii) If $L, L' \in \mathcal{L}$ are such that $L' \subsetneq L$, then

$$4\frac{\mu(L)}{|L|} < \frac{\mu(L')}{|L'|}.$$

Observe that the collections $\mathcal{Q}'(L)$ partition $\mathcal{Q}'$. A construction of the corona decomposition is accomplished via the following stopping-time argument from [1]. Let $\mathcal{L}_0$ consist of all $Q \in \mathcal{Q}'$ which are maximal for set inclusion. Recursively, $\mathcal{L}_{m+1}$ shall consist of all Q in the set

$$\bigcup_{L\in\mathcal{L}_m} \left\{ Q \in \mathcal{Q}' : Q \subset L \text{ and } \frac{\mu(Q)}{|Q|} > 4\frac{\mu(L)}{|L|} \right\}$$

for which Q is maximal for set inclusion, and $\mathcal{Q}'(L)$ is the collection of all $Q \in \mathcal{Q}'$ such that $Q \subset L$ and $Q \nsubseteq L'$ for any $L' \in \mathcal{L} := \bigcup_{m\geq 0} \mathcal{L}_m$ with $L' \subsetneq L$.

A straightforward consequence of the construction is

$$\Big| \bigcup_{\mathcal{L}\ni L'\subsetneq L} L' \Big| \leq \frac{1}{4}|L|, \; L \in \mathcal{L}. \tag{7}$$

To what extent does (7) still hold if we replace Lebesgue measure by the weight w? The following Lemma gives a partial answer to this question:

Lemma 8. *Let $\mathcal{L}$ be associated with the corona decomposition of an A_2 weight w. For any cube Q we have*

$$\sum_{\mathcal{L}\ni L\subset Q} w(L) \leq \frac{16}{9}[w]_{A_2} w(Q). \tag{8}$$

1.4 Idea of the proof of Theorem 5

In order to be able to apply Theorem 6, we start by noting that inequality (4) is equivalent to the following two weight version:

$$\|T(fw)\|_{L^2(w^{-1})} \lesssim [w]_{A_2}\|f\|_{L^2(w)}. \tag{9}$$

To verify (9), it is enough to show that the following conditions hold for all cubes Q, R of comparable size (*i.e.* such that $2^{-d\tau}|Q| \leq |R| \leq 2^{d\tau}|Q|$):

$$|\langle T(w1_Q), w^{-1}1_R\rangle| \lesssim [w]_{A_2} w(Q)^{1/2} w^{-1}(R)^{1/2}; \tag{10}$$

$$\int_Q |T(w1_Q)|^2 w^{-1} dx \lesssim [w]_{A_2}^2 w(Q). \tag{11}$$

The "weak boundedness" inequality (10) can be derived from the "T1" condition (11), and so we concentrate on verifying the latter. "Large scales" are easy to handle, and so we will limit ourselves to showing that

$$\Big\| \sum_{Q:Q\subset Q_0} \langle w, g_Q\rangle \gamma_Q \Big\|_{L^2(w^{-1})} \lesssim [w]_{A_2} w(Q_0)^{1/2}. \tag{12}$$

For cubes Q_0 and collections of cubes $\mathcal{Q}'$, we define two quantities:

$$H(Q_0, \mathcal{Q}') := \sum_{\mathcal{Q}'\ni Q\subset Q_0} \langle w, g_Q\rangle \gamma_Q;$$

$$H(\mathcal{Q}') := \sup_{Q_0\in\mathcal{Q}'} \frac{\|H(Q_0,\mathcal{Q}')\|_{L^2(w^{-1})}}{w(Q_0)^{1/2}}.$$

We pave the way to the corona decomposition by introducing the sets

$$\mathcal{Q}_n := \Big\{ Q \in \mathcal{Q} : 2^{n-1} < \frac{w(Q)}{|Q|}\frac{w^{-1}(Q)}{|Q|} \leq 2^n \Big\},$$

which essentially fix the local A_2 characteristic. Our goal is of course to show that $H(\mathcal{Q}) \lesssim [w]_{A_2}$. For that purpose, it is enough to show that $H(\mathcal{Q}_n) \lesssim 2^{n/2}[w]_{A_2}^{1/2}$.

Fix $Q_0 \in \mathcal{Q}_n$ which tests the supremum in the definition of $H(\mathcal{Q}_n)$. Let

$$\mathcal{P}_n := \{Q \in \mathcal{Q}_n : Q \subset Q_0\},$$

and consider the corona decomposition $(\mathcal{P}_n(L))_{L \in \mathcal{L}_n}$ of $\mathcal{P}_n$ with respect to the measure w (accomplished via the construction outlined in the previous section). Note that $\mathcal{L}_n \subset \mathcal{P}_n \subset \mathcal{Q}_n$.

We are seeking to prove

$$\|H(Q_0, \mathcal{Q}_n)\|^2_{L^2(w^{-1})} \lesssim 2^n [w]_{A_2} w(Q_0). \tag{13}$$

Since $H(Q_0, \mathcal{Q}_n) = \sum_{L \in \mathcal{L}_n} H(L, \mathcal{P}_n(L))$, the following Lemma will be useful in the proof of (13).

Lemma 9. *The following uniform distributional estimates hold for $L \in \mathcal{L}_n$:*

$$\Big|\Big\{x \in L : |H(L, \mathcal{P}_n(L))(x)| > Kt\frac{w(L)}{|L|}\Big\}\Big| \lesssim e^{-t}|L|; \tag{14}$$

$$w^{-1}\Big(\Big\{x \in L : |H(L, \mathcal{P}_n(L))(x)| > Kt\frac{w(L)}{|L|}\Big\}\Big) \lesssim e^{-t} w^{-1}(L). \tag{15}$$

The proof of Lemma 9 involves a further dyadic decomposition, and uses Proposition 4 and a version of John-Nirenberg inequality. I will omit the details and present them at the Summer School.

Letting $H_n(L) := |H(L, \mathcal{P}_n(L))|$, we have that

$$\begin{aligned}\|H(Q_0, \mathcal{Q}_n)\|^2_{L^2(w^{-1})} \leq & \Big\| \sum_{L \in \mathcal{L}_n} H_n(L) \Big\|^2_{L^2(w^{-1})} \\ = & \sum_{L \in \mathcal{L}_n} \|H_n(L)\|^2_{L^2(w^{-1})} + 2 \sum_{L \in \mathcal{L}_n} \sum_{\mathcal{L}_n \ni L' \subsetneq L} \int H_n(L) H_n(L') w^{-1} \\ =: & \, I + II.\end{aligned}$$

We estimate I (estimating II is similar but slightly more involved):

$$\begin{aligned}\|H_n(L)\|^2_{L^2(w^{-1})} \lesssim & \Big(\frac{w(L)}{|L|}\Big)^2 w^{-1}(L) \qquad \text{(by (15))} \\ = & w(L)\Big(\frac{w(L)}{|L|}\frac{w^{-1}(L)}{|L|}\Big) \\ \leq & 2^n w(L). \qquad \text{(since } L \in \mathcal{L}_n \subset \mathcal{Q}_n\text{)}\end{aligned}$$

The desired result now follows from Lemma 8:

$$I \lesssim 2^n \sum_{L \in \mathcal{L}_n} w(L) \lesssim 2^n [w]_{A_2} w(Q_0).$$

The author is indebted to Maria Carmen Reguera for a very useful discussion of some parts of [2] and to Michael Christ for valuable recommendations on the exposition.

References

[1] T. Hytönen, M. Lacey, M. C. Reguera, E. Sawyer, I. Uriarte-Truero and A. Vagharshakyan, *Weak and strong type A_p estimates for Calderón-Zygmund operators.* Preprint, arXiv:1006.2530v1.

[2] M. Lacey, S. Petermichl and M. C. Reguera, *Sharp A_2 inequality for Haar shift operators.* Math. Ann. **348** (2010), 127-141.

[3] F. Nazarov, S. Treil and A. Volberg, *Two weight inequalities for individual Haar multipliers and other well localized operators.* Math. Res. Lett. **15** (2008), no. 3, 583-597.

[4] S. Petermichl, *The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic.* Amer. J. Math. **129** (2007), no. 5, 1355-1375.

[5] E. M. Stein, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43. Princeton University Press, Princeton, NJ, 1993.

Diogo Oliveira e Silva, UC Berkeley
email: **dosilva@math.berkeley.edu**