WEIGHTED NORM THEORY

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Abstract. These notes are meant to provide a very short introduction to weighted norm theory. We present a well-known proof of a necessary and sufficient condition on a weight w for the maximal function to be bounded from $L^p(w)$ to $L^{p,\infty}(w)$. We also mention why the condition is enough to ensure (strong) boundedness on $L^p(w)$, and how to extend these results to various singular integral operators. We provide applications of the theory to elliptic PDE and harmonic analysis.

1. Basic theory

A **weight** w is an almost everywhere positive function which is locally integrable.

Our starting point will be the (uncentered) maximal function:

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(x)| dx,$$

where the supremum is taken over *all* cubes $Q \subset \mathbb{R}^n$ containing x. M is pointwise dominated by the Hardy-Littlewood maximal function, and is therefore bounded on $L^p(dx)$ for $1 < p \leq \infty$ and of weak type (1,1); boundedness on L^∞ is trivial, and the weak type (1,1) inequality is proved via a Vitali covering-type argument. The result follows by interpolation.

Fix[1] $1 < p < \infty$.

Question 1. *For which weights w is the maximal function M of strong type (p,p) with respect to the measure $w(x)dx$? More precisely, we are looking for necessary and sufficient conditions on w that will guarantee the existence of a constant $C < \infty$ such that*

$$\int_{\mathbb{R}^n} Mf(x)^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \tag{1}$$

for every $f \in L^p(w)$.

1.1. Necessary conditions.

The strong type inequality (1) implies a weak type inequality:

$$w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leq C\lambda^{-p} \int |f(x)|^p w(x) dx. \tag{2}$$

Date: September 26, 2010.

[1] Refer to the appendix for a brief discussion of the endpoints $p = 1, \infty$.

Let $f \in L^p(w)$ be such that $f \geq 0$ a.e. and choose a cube Q such that $f(Q) := \int_Q f > 0$. Let λ be such that $0 < \lambda < \frac{f(Q)}{|Q|}$. The first (trivial) observation is that $x \in Q$ implies $M(f1_Q)(x) > \lambda$. It follows that

$$w(Q) \leq w(\{x : M(f1_Q)(x) > \lambda\}) \leq C\lambda^{-p} \int_Q |f|^p w$$

for every $\lambda < \frac{f(Q)}{|Q|}$, and so

$$w(Q)\Big(\frac{f(Q)}{|Q|}\Big)^p \leq C \int_Q |f|^p w. \tag{3}$$

Specializing to $f = 1_S$ for some $S \subset Q$, we get that

$$w(Q)\Big(\frac{|S|}{|Q|}\Big)^p \leq Cw(S), \tag{4}$$

from which three conclusions can be drawn:

(i) The measure w is **doubling**: if $2Q$ denotes the cube with the same center as Q and twice the side length, then (4) implies that
$$w(2Q) \leq C2^{np}w(Q);$$

(ii) One also has that $w \in L^1_{loc}$ or $w = \infty$ a.e. (if there's a cube of infinite w-measure, the same is true for any larger cube, and so all sets of positive Lebesgue measure have infinite w-measure);

(i) Moreover, $w > 0$ a.e. or $w \equiv 0$ (if $w(S) = 0$ for some set S of positive Lebesgue measure, then $w(Q) = 0$ for every cube Q containing S).

In particular, the *a priori* assumptions on the weight w are actually a consequence of the weighted norm inequality.

The dual of $p \in (1, \infty)$ is, as usual, defined via the equation $\frac{1}{p} + \frac{1}{p'} = 1$; recall that $p' = \frac{p}{p-1}$ and $(p-1)(p'-1) = 1$. We call $\sigma := w^{1-p'}$ the **dual weight** of w. Letting $f = \sigma 1_Q$, equation (3) becomes

$$w(Q)\Big(\frac{\sigma(Q)}{|Q|}\Big)^p \leq C\sigma(Q).$$

Since the constant C is independent of the cube Q, we have that

$$\sup_Q \frac{w(Q)}{|Q|}\Big(\frac{\sigma(Q)}{|Q|}\Big)^{p-1} < \infty. \tag{5}$$

1.2. **Intermezzo.** The left-hand side of (5) is known as the A_p**-constant** of A_p-characteristic of w, commonly denoted by

$$[w]_{A_p} = \sup_Q \frac{w(Q)}{|Q|}\Big(\frac{\sigma(Q)}{|Q|}\Big)^{p-1} = \sup_Q \Big(\frac{1}{|Q|}\int_Q w\Big)\Big(\frac{1}{|Q|}\int_Q w^{1-p'}\Big)^{p-1}.$$

Not surprisingly, we let A_p be the set of all weights with finite A_p-constant. The following lemma contains some of the basic properties of A_p weights:

Lemma 2. *Let $1 < p_1, p_2, p < \infty$, $p^* = \max\{p_1, p_2\}$ and let w be a weight. Then, for $x \in \mathbb{R}^n$ and $\lambda > 0$,*

(i) $[w]_{A_p} \geq 1$;
(ii) $[\delta_\lambda w]_{A_p} = [\tau_x w]_{A_p} = [\lambda w]_{A_p} = [w]_{A_p}$;
(iii) $[w_1 + w_2]_{A_{p^*}} \leq [w_1]_{A_{p_1}} + [w_2]_{A_{p_2}}$;
(iv) $w \in A_p$ *iff* $\sigma \in A_{p'}$;
(v) $p < q \Rightarrow A_p \subset A_q$.

Assertion (i) is a direct consequence of Hölder inequality (write $1 = w^{1/p} w^{-1/p}$); both (i) and (ii) show that $[\cdot]_{A_p}$ is not a norm (even though it behaves like one in some sense, cf. (iii) – this is trivial, by the way: just note that $w_1 + w_2 \geq \max\{w_1, w_2\}$). To prove (iv), use $(p-1)(p'-1) = 1$. Finally, (v) is once again a consequence of Hölder: if $p < q$, then $p' > q'$ and

$$\Big(\frac{1}{|Q|}\int_Q (w^{-1})^{q'-1}\Big)^{q-1} \leq \Big(\frac{1}{|Q|}\int_Q (w^{-1})^{p'-1}\Big)^{p-1}$$

since $L^{p'-1}(Q) \subset L^{q'-1}(Q)$.

Example 3. *The prime example of A_p weights are power weights (easy to compute with, too). It is a straightforward exercise to show that, for $1 < p < \infty$ and $\gamma \in \mathbb{R}$,*

$$w(x) = |x|^\gamma \in A_p(\mathbb{R}^n) \text{ iff } -n < \gamma < n(p-1).$$

To see that the range is sharp, observe that outside of it $|x|^\gamma$ and $|x|^{\gamma(1-p')}$ are not both locally integrable. Note however that w is doubling in the larger range $\gamma > -n$; in particular, the property of being doubling gives an interesting insight into the nature of A_p weights, but does not capture the full essence of the matter.

1.3. **Sufficient conditions (for the weak type inequality).** Our first goal is to prove that, if $w \in A_p$, then w satisfies (3) and therefore (4). We use Hölder (and the A_p condition, naturally):

$$\begin{aligned}[f(Q)]^p = \Big(\int_Q w^{-1/p} f w^{1/p}\Big)^p &\leq \Big(\int_Q w^{-p'/p}\Big)^{p/p'} \Big(\int_Q |f w^{1/p}|^p\Big) \\ &= \Big(\frac{1}{|Q|}\int_Q w^{1-p'}\Big)^{p-1} |Q|^{p-1} \Big(\int_Q |f|^p w\Big) \\ &\leq [w]_{A_p} \Big(\frac{1}{|Q|}\int_Q w\Big)^{-1} |Q|^{p-1} \Big(\int_Q |f|^p w\Big).\end{aligned}$$

Rearranging, we get (3) with $C = [w]_{A_p}$.

We now show that M satisfies a weighted weak type (p,p) inequality. Let $f \in L^p(w)$. Without loss of generality we may assume that $f \geq 0$ and $f \in L^1$ (otherwise replace f by $f 1_{B(0,k)}$ and note that the argument yields bounds independent of k).

Calderón-Zygmund decompose f at height $4^{-n}\lambda$. This yields a decomposition $f = g + b$, where the "good" part g is bounded by the height, $\|g\|_\infty \leq 4^{-n}\lambda$, and the "bad" part $b = \sum_j b_j$ is supported on a union of disjoint cubes, $supp(b_j) \subset Q_j$, and satisfies a useful cancellation condition, $\int_{Q_j} b_j = 0$; moreover, the size of the cubes satisfies a favorable bound, $\sum_j |Q_j| \leq (4^{-n}\lambda)^{-1}\|f\|_1$. This is accomplished via a stopping-time argument that forces

$$4^{-n}\lambda < \frac{f(Q_j)}{|Q_j|} \leq 2^n(4^{-n}\lambda) = 2^{-n}\lambda, \text{ for every } j. \tag{6}$$

The crucial observation is the following:

$$\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \subset \bigcup_j 3Q_j. \tag{7}$$

To see why (7) is true, let $x \notin \bigcup_j 3Q_j$. Let Q be any cube containing x of side length $l(Q) \sim 2^{k-1}$. There are at most 2^n dyadic cubes $(R_i)_i \subset \mathcal{Q}_k$ that intersect Q; none of them can be contained in any stopping cube Q_j, otherwise $x \in \bigcup_j 3Q_j$ (draw picture). This means that the average of f over any of the R_i is $< 4^{-n}\lambda$, and this forces

$$\frac{1}{|Q|}\int_Q f = \sum_i \frac{|R_i|}{|Q|}\frac{1}{|R_i|}\int_{Q\cap R_i} f \leq \lambda.$$

It follows that:

$$\begin{aligned}
w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) &\leq w\Big(\bigcup_j 3Q_j\Big) \\
&\leq \sum_j w(3Q_j) \\
&\leq C\sum_j 3^{np} w(Q_j) \qquad \text{by (4)} \\
&\leq C3^{np}\sum_j \Big(\frac{|Q_j|}{f(Q_j)}\Big)^p \int_{Q_j} f^p w \qquad \text{by (3)} \\
&\leq C3^{np}(4^{-n}\lambda)^{-p}\int f^p w \qquad \text{by (6)} \\
&\lesssim_{n,p} \lambda^{-p}\|f\|^p_{L^p(w)}.
\end{aligned}$$

2. WEIGHTS AND STRONG TYPE INEQUALITIES

We have seen that the A_p condition (5) is necessary and sufficient for the weak type inequality (2) to hold. That the A_p condition is the right one to consider is shown by the following result of Muckenhoupt [6]:

Theorem 4. *If $1 < p < \infty$ then M is bounded on $L^p(w)$ if and only if $w \in A_p$.*

The key ingredient to prove Theorem 4 is the following "reverse Hölder inequality"[2], which is the deepest and most significant part of the whole theory:

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[2] The name comes from the fact that its converse is an immediate consequence of Hölder inequality.

Proposition 5. *Let* $1 < p < \infty$ *and* $w \in A_p$*. Then there exist constants* C *and* $\epsilon > 0$*, depending only on* p *and* $[w]_{A_p}$*, such that for any cube* Q*,*

$$\Big(\frac{1}{|Q|}\int_Q w^{1+\epsilon}\Big)^{\frac{1}{1+\epsilon}} \leq \frac{C}{|Q|}\int_Q w. \tag{8}$$

Somewhat informally, this proposition states that on average, the values of an A_p weight do not fluctuate too much - note that inequality (8) is trivial for weights w that are bounded from above and from below.

The proof involves a stopping-time argument similar to the one used to prove John-Nirenberg inequality[3], where a sequence of CZ decompositions at increasing heights is performed.

We will omit the details and focus instead on the following surprising consequence, which says that A_p classes are open in the sense that if $w \in A_p$, then $w \in A_q$ for some $q < p$:

Corollary 6. $A_p = \bigcup_{q<p} A_q \quad (1 < p < \infty)$.

Proof. The interesting inclusion is $\subset$, since the other one is an immediate consequence of Lemma 2 (v). Let $w \in A_p$. Then $\sigma = w^{1-p'} \in A_{p'}$ satisfies a reverse Hölder inequality:

$$\Big(\frac{1}{|Q|}\int_Q w^{(1-p')(1+\epsilon)}\Big)^{\frac{1}{1+\epsilon}} \leq \frac{C}{|Q|}\int_Q w^{1-p'}.$$

Define q by $(1-p')(1+\epsilon) = 1-q'$. Check that $q < p$ and that this last inequality is equivalent to

$$\Big(\frac{1}{|Q|}\int_Q w^{1-q'}\Big)^{q-1} \leq C\Big(\frac{1}{|Q|}\int_Q w^{1-p'}\Big)^{p-1}.$$

This easily implies that $w \in A_q$ (multiply both sides of the inequality by $\frac{1}{|Q|}\int_Q w$). □

The proof of Theorem 4 is now immediate. Start by noting that $L^\infty(w) = L^\infty$ with equality of norms since $w(E) = 0$ iff $|E| = 0$. In particular, M is (trivially) bounded on $L^\infty(w)$. If $w \in A_p$, then $w \in A_q$ for some $q < p$. By the results of the previous section, M is of weak type (q, q). By (Marcinkiewicz) interpolation, M is bounded on $L^p(w)$.

3. Extrapolation and weighted inequalities for singular integrals

The following extrapolation theorem of Rubio de Francia [4] bears some resemblance to Calderón and Zygmund's celebrated result on singular integral operators:

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[3] One of the forms of this celebrated inequality states that if $f \in BMO$, that is, if

$$\|f\|_* := \sup_Q \frac{1}{|Q|}\int_Q \Big|f - \frac{f(Q)}{|Q|}\Big| < \infty,$$

then there exists $c = c(n) < \infty$ such that

$$|\{x \in Q : \Big|f(x) - \frac{f(Q)}{|Q|}\Big| > \lambda\}| \lesssim_n e^{-c\lambda\|f\|_*^{-1}}|Q| \quad \text{for any cube } Q \text{ and } \lambda > 0.$$

Theorem 7. *Let $1 < r < \infty$. If T is a bounded operator on $L^r(w)$ for any $w \in A_r$ (with operator norm depending only on $[w]_{A_r}$), then T is bounded on $L^p(w)$ for any $p \in (1,\infty)$ and $w \in A_p$ (with operator norm depending only on $[w]_{A_p}$).*

Overcoming some serious technical difficulties, Hunt, Muckenhoupt and Wheeden [5] were able to extend Theorem 4 to the case of the Hilbert transform[4]:

Theorem 8. *The Hilbert transform is bounded on $L^p(w)$ if and only if $w \in A_p$.*

Coifman and Fefferman [2] greatly simplified the proof of Theorem 8 and extended the theory to (regular) Calderón-Zygmund operators.

4. Appendix

4.1. A_1 and A_∞.

We have limited our discussion to the case $1 < p < \infty$ since the A_p condition looks somewhat different at the endpoints $p = 1, \infty$.

If $p = 1$, then equation (4) becomes

$$\frac{w(Q)}{|Q|} \leq C\frac{w(S)}{|S|}.$$

By Lebesgue differentiation theorem (recall that $w \in L^1_{loc}$),

$$\frac{w(Q)}{|Q|} \leq Cw(x) \quad \text{for a.e. } x \in Q.$$

This is the A_1 condition. A_1 weights show up in factorization theorems, which reveal how A_p weights ($p > 1$) can be fashioned from A_1 weights. As an example, here is a result of P. Jones [9]:

Theorem 9. *Suppose that w_1 and w_2 are A_1 weights. If $1 \leq p < \infty$, then $w = w_1 w_2^{1-p}$ belongs to A_p. Conversely, suppose that w is an A_p weight. Then there exist w_1 and w_2 in A_1 such that $w = w_1 w_2^{1-p}$.*

When $p = 2$, this statement is particularly elegant: a weight is in A_2 precisely when it is the quotient of two A_1 weights.

A weight w is said to be in A_∞ if there exists $\delta > 0$ such that, given a cube Q and a subset $S \subset Q$,

$$\frac{w(S)}{w(Q)} \leq C\Big(\frac{|S|}{|Q|}\Big)^{\delta}. \tag{9}$$

This (together with (4)) constitutes a "fairness" principle governing A_p weights: $\frac{w(S)}{w(Q)}$ is bounded below and above by quantities that depend only on the ratio $\frac{|S|}{|Q|}$. That any weight

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[4] Recall its definition:

$$Hf(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

$w \in A_p$ (for $p < \infty$) satisfies (9) is a consequence of the reverse Hólder inequality (8), see [3].

The following result of Muckenhoupt shows that Corollary 6 remains true at the endpoint $p = \infty$:

Theorem 10. $A_\infty = \bigcup_{q<\infty} A_q$.

A_∞ weights are useful in BMO theory. For instance, if $w \in A_\infty$, then $\log w \in BMO$ (see [4]). This is another instance of the principle that the average fluctuation of the magnitude of w on every ball is uniformly controlled, and gives a particularly elegant way to prove the well-known fact that $\log|P| \in BMO$ whenever P is a polynomial.

4.2. Applications to PDE.

The Beltrami equation on the plane is

$$f_z - \mu f_{\bar{z}} = 0,$$

where μ is a bounded measurable function, $\|\mu\|_\infty = k < 1$. The question is: what is the minimal requirement of the type $f \in W^{1,q}_{loc}$ that guarantees that any solution of the Beltrami equation with any $\|\mu\|_\infty = k < 1$ is a continuous function? A deep result of Astala asserts that $f \in W^{1,1+k+\epsilon}_{loc}$ suffices if $\epsilon > 0$. On the other hand, Lehto and Iwaniec showed that $q < 1 + k$ is not sufficient. The borderline case $q = 1 + k$ was solved by Petermichl and Volberg [8], who showed that the solution is still continuous in that case. The problem had been reduced to a question concerning sharp weighted bounds for the Beurling-Ahlfors[5] transform in $L^p(w)$. It is interesting to note that the exact sharp growth of the Beurling-Ahlfors transform (in terms of $[w]_{A_p}$) was needed, while the other cases could be handled merely using *any* bound for the operator in the weighted space.

4.3. Applications to harmonic analysis.

There is an intimate connection between weights, BMO functions and Fefferman's duality theorem[6], which can be roughly described by saying that BMO consists of logarithms of "good" weights. As previously mentioned, if $w \in A_\infty$, then $\log w \in BMO$. One actually has that [4]:

$$BMO = \{\alpha \log w : \alpha \geq 0, w \in A_p\}.$$

As a last remark, observe that if we replace M by the Fourier transform in a 2-weight version of the strong-type inequality (1), we obtain

$$\int_{S^{n-1}} |\widehat{f}(\xi)|^q w(\xi) d\xi \lesssim \int_{\mathbb{R}^n} |f(x)|^p dx,$$

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[5] Recall its definition: for $f \in L^p(\mathbb{C})$ and $1 \leq p < \infty$,

$$Sf(z) = -\frac{1}{\pi} p.v. \int_{\mathbb{C}} \frac{f(w)}{(z-w)^2} dw,$$

where dw denotes 2-dimensional Lebesgue measure on $\mathbb{C}$. As a singular integral operator of CZ type, S is bounded on L^p if $1 < p < \infty$ and of weak-type (1,1).

[6] $(H^1)^* = BMO$.

where w denotes the surface measure on $S^{n-1} \subset \mathbb{R}^n$. This is the famous restriction problem, probably one of the most interesting and hard problems concerning weighted inequalities.

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