Sharp A_2 inequality for Haar shift operators

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Lemma

For $x \in \mathbb{R}^n$ and $\lambda > 0$,

(i) $[w]_{A_p} \geq 1$**;**

(ii) $[\delta_\lambda w]_{A_p} = [\tau_x w]_{A_p} = [\lambda w]_{A_p} = [w]_{A_p}$**;**

(iii) $[w_1 + w_2]_{A_p} \leq [w_1]_{A_p} + [w_2]_{A_p}$**;**

(iv) $w \in A_p$ **iff** $w^{1-p'} \in A_{p'}$**;**

(v) $p < q \Rightarrow A_p \subset A_q$**.**

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$$\int |Hf(x)|^2 w(x) dx \leq C \int |f(x)|^2 w(x) dx$$

iff

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- How does C depend on $[w]_{A_2}$?

Earlier results

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$$\|Hf\|_{L^2(w)} \lesssim [w]_{A_2} \|f\|_{L^2(w)}.$$

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- Tools: Bellman function techniques, Haar shift operators.
- Application: borderline regularity of solutions of the Beltrami equation.

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Definition

A **Haar function on a cube** Q is a function h_Q supported on Q such that:

(i) h is constant on dyadic subcubes of Q;

(ii) cancellation: $\int_Q h_Q = 0$;

(iii) size: $\|h_Q\|_\infty \leq |Q|^{-1/2}$.

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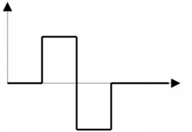

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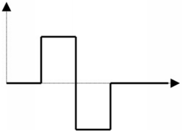

Consequence: Haar functions are L^2 normalized.

Haar shift operators

Definition

We say that T is a **Haar shift operator of index** $\tau \in \mathbb{N}_0$ **on** $\mathbb{R}^d$ if

$$Tf = \sum_{Q \in \mathcal{Q}} \langle f, g_Q \rangle \gamma_Q,$$

where

$$g_Q, \gamma_Q \in span\{h_{Q'} : Q' \subset Q, 2^{-\tau d}|Q| \leq |Q'|\}, \text{ and}$$

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These are once again *cancellation* and *size* conditions, respectively.

Two examples

Example

(**Multipliers**, index 0) Given a bounded sequence $\alpha = (\alpha_Q)_{Q \in \mathcal{Q}}$, consider the Haar multiplier T^α given by

$$T^\alpha f = \sum_{Q \in \mathcal{Q}} \alpha_Q \langle f, h_Q \rangle h_Q.$$

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Example

(**Haar shifts**, index 1) For a one-dimensional example, consider the Haar shift S given by

$$Sf = \sum_{I\in\mathcal{Q}} \langle f, h_I\rangle (h_{I_r} - h_{I_l}).$$

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Let T be a Haar shift operator of index τ on $\mathbb{R}^d$. Then T is bounded on $L^2(dx)$ with norm $\lesssim \tau$, and T maps $L^1(dx)$ into $L^{1,\infty}(dx)$ with norm $\lesssim 2^{\tau d}$.

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(L^2) *Consider the operator at scale 2^s:*

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($L^{1,\infty}$) *CZ decomposition ($x \notin Q^{(\tau)} \Rightarrow T(1_Q b)(x) = 0$).*

The Goal

Theorem

Let T be a Haar shift operator of index τ on $\mathbb{R}^d$, and let w be an A_2 weight. Then

$$\|T\|_{L^2(w)\to L^2(w)} \lesssim_{d,\tau} [w]_{A_2}.$$

1. A two weight $T1$ theorem

Theorem

(Nazarov, Treil and Volberg '08) **Let T be a Haar shift operator of index τ on $\mathbb{R}^d$, and let σ and μ be two positive measures. The inequality**

$$\|T(\sigma f)\|_{L^2(\mu)} \lesssim \|f\|_{L^2(\sigma)}$$

holds if and only if there exist constants $C_1, C_2, C_3 < \infty$ such that for all cubes Q, Q', Q'' with $Q', Q'' \subset Q$ and $2^{-\tau d}|Q| \leq |Q'|, |Q''|$,

$$\left| \int_{Q''} T(\sigma 1_{Q'}) d\mu \right| \leq C_1 \sigma(Q')^{1/2} \mu(Q'')^{1/2},$$

$$\|T(\sigma 1_Q)\|_{L^2(Q,\mu)} \leq C_2 \sigma(Q)^{1/2} \text{ and}$$

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- As a corollary of the proof, one gets that

$$\|T(\sigma \cdot)\|_{L^2(\sigma) \to L^2(\mu)} \lesssim C_1 + C_2 + C_3.$$

2. A corona decomposition

Definition

Let $\mathcal{Q}' \subset \mathcal{Q}$ be any bounded collection of cubes, and let μ be a positive measure. Let $\mathcal{L} \subset \mathcal{Q}'$. We call $(\mathcal{Q}'(L))_{L\in\mathcal{L}}$ a **corona decomposition of $\mathcal{Q}'$ with respect to μ** if the following conditions hold:

(i) For every $Q \in \mathcal{Q}'$, there exists $L \in \mathcal{L}$ such that $Q \subset L$. Let $\lambda(Q) \in \mathcal{L}$ denote the minimal cube which contains Q, and set $\mathcal{Q}'(L) := \{Q \in \mathcal{Q}' : \lambda(Q) = L\}$. Then

$$\frac{\mu(Q)}{|Q|} \leq 4\frac{\mu(\lambda(Q))}{|\lambda(Q)|};$$

(ii) If $L, L' \in \mathcal{L}$ are such that $L' \subsetneq L$, then

$$4\frac{\mu(L)}{|L|} < \frac{\mu(L')}{|L'|}.$$

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for which Q is maximal for set inclusion, and $\mathcal{Q}'(L)$ is the collection of all $Q \in \mathcal{Q}'$ such that $Q \subset L$ and $Q \not\subseteq L'$ for any $L' \in \mathcal{L} := \bigcup_{m \geq 0} \mathcal{L}_m$ with $L' \subsetneq L$.

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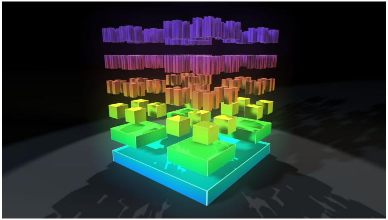
- 

Properties of the corona $(\mathcal{Q}'(L))_{L\in\mathcal{L}}$

Lemma 1

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Lemma 2

Let $\mathcal{L}$ be associated with the corona decomposition of an A_2 weight w. For any cube Q we have

$$\sum_{\mathcal{L}\ni L\subset Q} w(L) \le \frac{16}{9}[w]_{A_2} w(Q).$$

Proof of Lemma 2

It is enough to show

$$w\Big(\bigcup_{\mathcal{L}\ni L'\subsetneq L} L'\Big) \le \Big(1 - \frac{9}{16}[w]_{A_2}^{-1}\Big) w(L).$$

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$$\frac{|E|}{|L|} = \frac{1}{|L|}\int_E w^{1/2} w^{-1/2} \le \Big(\frac{w(E)}{|L|}\frac{w^{-1}(L)}{|L|}\Big)^{1/2} \le \Big([w]_{A_2}\frac{w(E)}{w(L)}\Big)^{1/2}.$$

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Some reductions

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is equivalent to the following two weight version:

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By the two weight $T1$ theorem, ETS that

- $|\langle T(w1_Q), w^{-1}1_R\rangle| \lesssim [w]_{A_2} w(Q)^{1/2} w^{-1}(R)^{1/2}$;

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- $\int_Q |T(w1_Q)|^2 w^{-1} dx \lesssim [w]_{A_2}^2 w(Q)$

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$$\text{Goal: } \Big\| \sum_{Q:Q\subset Q_0} \langle w, g_Q\rangle \gamma_Q \Big\|_{L^2(w^{-1})} \lesssim [w]_{A_2} w(Q_0)^{1/2}.$$

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- New goal: $H(\mathcal{Q}) \lesssim [w]_{A_2}$ $\Big(\Leftarrow H(\mathcal{Q}_n) \lesssim 2^{n/2}[w]_{A_2}^{1/2} \Big)$.

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$$H(\mathcal{Q}_n) := \sup_{Q_0 \in \mathcal{Q}_n} \frac{\|H(Q_0, \mathcal{Q}_n)\|_{L^2(w^{-1})}}{w(Q_0)^{1/2}};$$

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- Organize! $H(Q_0, \mathcal{Q}_n) = \sum_{L \in \mathcal{L}_n} H(L, \mathcal{P}_n(L))$.

Key estimates

Lemma

The following distributional estimates hold uniformly for $L \in \mathcal{L}_n$:

$$\left|\left\{x \in L : |H(L, \mathcal{P}_n(L))(x)| > Kt\frac{w(L)}{|L|}\right\}\right| \lesssim e^{-t}|L|;$$

$$w^{-1}\left(\left\{x \in L : |H(L, \mathcal{P}_n(L))(x)| > Kt\frac{w(L)}{|L|}\right\}\right) \lesssim e^{-t}w^{-1}(L).$$

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Proof.

On Thursday. ☐

New goal: $\|H(Q_0, \mathcal{Q}_n)\|^2_{L^2(w^{-1})} \lesssim 2^n [w]_{A_2} w(Q_0)$.

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$$\begin{aligned}
\|H(Q_0, \mathcal{Q}_n)\|^2_{L^2(w^{-1})} &\leq \Big\| \sum_{L \in \mathcal{L}_n} H_n(L) \Big\|^2_{L^2(w^{-1})} \\
&= \sum_{L \in \mathcal{L}_n} \|H_n(L)\|^2_{L^2(w^{-1})} + 2 \sum_{L \in \mathcal{L}_n} \sum_{\mathcal{L}_n \ni L' \subsetneq L} \int H_n(L) H_n(L') w^{-1} \\
&=: I + II.
\end{aligned}$$

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\end{aligned}$$

We'll estimate I and leave II in suspense.

Estimating I

$$\begin{aligned}
\|H_n(L)\|^2_{L^2(w^{-1})} =& 2\int_0^\infty s\cdot w^{-1}(\{x\in L: |H(L,\mathcal{P}_n(L))(x)|>s\})ds \\
\lesssim& \Big(\frac{w(L)}{|L|}\Big)^2 w^{-1}(L) \qquad \text{(by key lemma)} \\
=& w(L)\Big(\frac{w(L)}{|L|}\frac{w^{-1}(L)}{|L|}\Big) \\
\leq& 2^n w(L). \qquad \text{(since } L\in\mathcal{L}_n\subset\mathcal{Q}_n\text{)}
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The result now follows from Lemma 2:

Estimating I

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=& w(L)\Big(\frac{w(L)}{|L|}\frac{w^{-1}(L)}{|L|}\Big) \\
\le& 2^n w(L). \qquad \text{(since } L\in\mathcal{L}_n\subset\mathcal{Q}_n\text{)}
\end{aligned}$$

The result now follows from Lemma 2:

$$I := \sum_{L\in\mathcal{L}_n} \|H_n(L)\|^2_{L^2(w^{-1})} \lesssim 2^n \sum_{L\in\mathcal{L}_n} w(L) \lesssim 2^n [w]_{A_2} w(Q_0).$$

Next time

- Proof of key lemma;

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- Estimate II.