THE RESTRICTION PROBLEM

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ABSTRACT. E. M. Stein's restriction problem for the Fourier transform is a deep and only partially solved conjecture in harmonic analysis. We state the problem and the Tomas-Stein theorem, which solves a particularly useful case of the conjecture. We give a proof of the theorem, and then turn to the existence of extremizers for the corresponding inequality.

1. INTRODUCTION

For a function $f \in L^1(\mathbb{R}^n)$, we can define its Fourier transform:

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx.$$

It is a well known fact that $\widehat{f}$ is continuous and decays to 0 as $|\xi| \to \infty$. In particular, it is defined everywhere. On the other hand, the Fourier transform of an L^2 function is itself no better than an L^2 function, and so can only be defined almost everywhere and is thus completely arbitrary on sets of measure zero.

In the late 60's Stein asked whether it is possible to restrict the Fourier transform $\widehat{f}$ of a function $f \in L^p(\mathbb{R}^n)$ with $p \in [1, 2]$ to the sphere S^{n-1} as a function in $L^q(S^{n-1})$ for some $q \in [1, \infty]$. In other words, is there a bound

$$\|\widehat{f}\,|_{S^{n-1}}\,\|_{L^q(S^{n-1})} \leq C\|f\|_{L^p(\mathbb{R}^n)} \tag{1}$$

for every[1] $f \in \mathcal{S}(\mathbb{R}^n)$, with a constant $C = C(n, p, q)$?

(i) Example: $p = 1, q = \infty, C = 1$;
(ii) Non-example: $p = 2$.

Instead of studying the restriction operator, we'll study its dual (sometimes called the extension operator). We are thus able to rephrase (1) in terms of the surface measure σ on the sphere: since

$$\int_{S^{n-1}} \widehat{f}(\xi)g(\xi)d\sigma(\xi) = \int_{S^{n-1}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)g(\xi)dxd\sigma(\xi) = \int_{\mathbb{R}^n} f(x)\widehat{g\sigma}(x)dx,$$

if $Tf := \widehat{f}\,|_{S^{n-1}}$, then $T^*f = \widehat{f\sigma}$.

That this is indeed the same problem follows from the following lemma:

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[1]This is enough since the inclusion $\mathcal{S} \subset L^1 + L^2$ is dense.

Lemma 1. *Let μ be a finite positive measure. The following are equivalent for any q and any C:*

(1) $\|\widehat{f\mu}\|_q \leq C\|f\|_{L^2(d\mu)}$, *for every* $f \in L^2(d\mu)$;
(2) $\|\widehat{g}\|_{L^2(d\mu)} \leq C\|g\|_{q'}$, *for every* $g \in \mathcal{S}(\mathbb{R}^n)$;
(3) $\|\widehat{\mu} * f\|_q \leq C^2\|f\|_{q'}$, *for every* $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Lemma 1 fits into the abstract setup:

$$T : L^{q'} \to L^2 \Leftrightarrow T^* : L^2 \to L^q \Leftrightarrow T^*T : L^{q'} \to L^q.$$

If one prefers more down-to-earth computations, I suggest the following:

(1) $\Leftrightarrow$ (2): Let $g \in \mathcal{S}$, $f \in L^2(d\mu)$. Then:

$$\int \widehat{g} f d\mu = \int \widehat{f\mu} \cdot g dx.$$

(2) $\Leftrightarrow$ (3): If $f, g \in \mathcal{S}$ and μ is (say) compactly supported, then:

$$\int \widehat{f}\widehat{\overline{g}} d\mu = \int (\hat{\mu} * \overline{g}) f dx.$$

□

Given a function $f : S^{n-1} \to \mathbb{C}$, we will consider its Fourier transform

$$\widehat{f\sigma}(\xi) = \int_{S^{n-1}} f(x) e^{-ix\cdot\xi} d\sigma(x).$$

The main result that will be discussed in these notes dates back to the 70's:

Theorem 2. *(Tomas-Stein) If $f \in L^2(S^{n-1})$, then*

$$\|\widehat{f\sigma}\|_q \leq C_{n,q}\|f\|_{L^2(S^{n-1})} \text{ for } q \geq \frac{2(n+1)}{n-1}. \tag{2}$$

Some remarks are in order:

Remark 3. *The assumption on q is of the form $q \geq q_0$. The reason for this is that there is an obvious estimate*

$$\|\widehat{f\sigma}\|_\infty \leq \|f\|_{L^1(S^{n-1})} \lesssim \|f\|_{L^2(S^{n-1})},$$

and it follows from Riesz-Thörin that if (2) holds for a given q, then it also holds for any larger q.

Remark 4. *We will see below that the range of q in (2) is the best possible, but Theorem 2 does not completely solve the restriction problem. The following conjecture of Stein is still open[2]: Prove that if $f \in L^\infty(S^{n-1})$, then*

$$\|\widehat{f\sigma}\|_q \leq \|f\|_{L^\infty(S^{n-1})}, \tag{3}$$

[2] A solution of the restriction conjecture would have dramatic implications in analysis, as Tao's diagram illustrates. An example is the Bochner-Riesz conjecture ([3], p.189): if for some $p > 1$,

$$\|\widehat{f}\,|_{S^{n-1}}\,\|_{L^2(S^{n-1})} \lesssim \|f\|_p,$$

for all $q > \frac{2n}{n-1}$. Note that (3) is true if $f \equiv 1$ (see remark 6(i)). Of course there is a difference in the L^q exponent in (2) and the one which is conjectured for L^∞ densities. Until the work of Bourgain in the early 90's it was unknown (for $n \geq 3$) whether the estimate (3) was true even for some q less than the Tomas-Stein exponent $\frac{2(n+1)}{n-1}$.

Remark 5. *(Knapp's example) This shows that $q \geq \frac{2(n+1)}{n-1}$ is the best possible in (2). In order to distinguish between L^2 and L^∞ norms one should use a highly localized function. One the other hand, the uncertainty principle[3] asserts that if a measure μ is supported on an ellipsoid E, then for many purposes $\widehat{\mu}$ may be regarded as being almost constant on any dual[4] ellipsoid E^*. We explore this principle by taking $f \equiv$ characteristic function of a rectangle "adapted" to the sphere. Let*

$$C_\delta := \{x \in S^{n-1} : 1 - x \cdot e_n \leq \delta^2\}.$$

Since $|x - e_n|^2 = 2(1 - x \cdot e_n)$,

$$x \in C_\delta \Leftrightarrow |x - e_n| \leq C\delta$$

for an appropriate constant C (draw picture). We let $f := \chi_{C_\delta}$ and calculate $\|f\|_{L^2(S^{n-1})}$ and $\|\widehat{f\sigma}\|_q$.

The first one is easy:

$$\|f\|_{L^2(S^{n-1})} = |C_\delta|^{1/2} \sim \delta^{\frac{n-1}{2}}.$$

As for the second one, start by observing that supp$(f\sigma)$ is contained in the rectangle R_δ centered at e_n with length $\sim \delta^2$ in the e_n direction and length $\sim \delta$ in the remaining $n-1$ orthogonal directions. According to the heuristics discussed above, one should look at $\widehat{f\sigma}$ on the dual rectangle R^*_δ centered at 0: suppose that, say, $|\xi_n| \leq C_1^{-1}\delta^{-2}$ and $|\xi_j| \leq C_1^{-1}\delta^{-1}$

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then T^a is bounded on L^p for all a such that $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{2a+1}{2n}$ holds. Recall that the Bochner-Riesz multipliers are defined for $a > 0$ in the following way:

$$(T^a f)^\wedge(\xi) = (1 - |\xi|^2)_+^a \widehat{f}(\xi).$$

[3]The simplest rigorous statement of this principle is the L^2 Bernstein inequality: if $f \in L^2$ and supp$(\widehat{f}) \subset B(0,R)$, then $f \in C^\infty$ and there is an estimate

$$\|D^\alpha f\|_2 \leq R^{|\alpha|}\|f\|_2.$$

[4]Let $E \subset \mathbb{R}^n$ be an ellipsoid, that is, a set of the form

$$E = \{x \in \mathbb{R}^n : \sum_j \frac{|(x-a)\cdot e_j|^2}{r_j^2} \leq 1\}.$$

An ellipsoid E^* is said to be **dual to** E if E^* has the same axes as E and reciprocal axis lengths.

if $j < n$. Then

$$\begin{aligned}|\widehat{f\sigma}(\xi)| =&\Big|\int_{S^{n-1}} f(x)e^{-ix\cdot\xi}d\sigma(x)\Big|\\ =&\Big|\int_{C_\delta} e^{-i(x-e_n)\cdot\xi}d\sigma(x)\Big|\\ \geq&\int_{C_\delta}\cos((x-e_n)\cdot\xi)d\sigma(x).\end{aligned}$$

If C_1 is large enough then $|(x-e_n)\cdot\xi| \leq \frac{\pi}{3}$ for every $x \in C_\delta$ and $\xi \in R_\delta^*$. If follows that

$$|\widehat{f\sigma}(\xi)| \geq \frac{1}{2}|C_\delta|$$

if $\xi \in R_\delta^*$, which itself has volume $\sim \delta^{-(n+1)}$. All in all,

$$\begin{aligned}\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^n)} \geq&\Big(\int_{R_\delta^*}|\widehat{f\sigma}(\xi)|^q d\xi\Big)^{1/q}\\ \gtrsim&|C_\delta||R_\delta^*|^{1/q} \sim \delta^{n-1}\delta^{-\frac{(n+1)}{q}}.\end{aligned}$$

So if (2) is to hold then

$$\delta^{n-1-\frac{n+1}{q}} \lesssim \delta^{\frac{n-1}{2}}$$

uniformly in $\delta \in (0,1]$. Hence $n-1-\frac{n+1}{q} \geq \frac{n-1}{2}$ i.e. $q \geq \frac{2(n+1)}{n-1}$, as desired.

Remark 6. *(The role of curvature)*

1. A key-ingredient in the proof of Theorem 2 will be the fact that the (C^∞) function $\widehat{\sigma}$ satisfies

$$|\widehat{\sigma}(\xi)| \lesssim (1+|\xi|)^{-\frac{(n-1)}{2}} \tag{4}$$

for large ξ. In particular, $\widehat{\sigma} \in L^p(\mathbb{R}^n)$ iff $p > \frac{2n}{n-1}$.

The estimate (4) follows from the rotational symmetry of S^{n-1} (in particular, $\widehat{\sigma}(\xi) = \widehat{\sigma}(-\xi) = \check{\sigma}(\xi)$) and stationary phase:

$$\widehat{\sigma}(\lambda e_n) = \int_{S^{n-1}} e^{i\lambda x_n}d\sigma(x) = \int_{\mathbb{R}^{n-1}} e^{i\lambda(1-|x'|^2)^{1/2}}(1-|x'|^2)^{-1/2}\eta(x')dx'$$

and $(1-|x'|^2)^{1/2} = 1 - \frac{|x'|^2}{2} + O(|x'|^4)$ has a nondegenerate critical point[5] at $x' = 0$. Alternatively, recall that

$$\widehat{\sigma}(\xi) = |\xi|^{\frac{2-n}{2}}J_{(n-2)/2}(|\xi|)$$

and look up asymptotics of Bessel functions[6].

[5]Note that nonvanishing Gaussian curvature of a manifold given be a (global) graph parametrization $\phi(u) = (u, f(u))$ is equivalent to $\det Hess(f) \neq 0$.

[6]Recall the relevant definitions (p.338, [7]): the Bessel function $J_m(r)$ of integral order m is defined by

$$J_m(r) := \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin\theta}e^{-im\theta}d\theta.$$

2. If $\Sigma \subset \mathbb{R}^n$ is an hyperplane with induced Lebesgue measure σ_Σ, then $\widehat{\sigma_\Sigma}$ has no decay in the normal direction to Σ. For another instance of this phenomenon, consider the function

$$f(x) := \frac{\psi(x_2, \ldots, x_n)}{1 + |x_1|}$$

where ψ is a bump function. Then $f \in L^p(\mathbb{R}^n)$ for every $p > 1$ but $\widehat{f}(\xi) = \infty$ whenever $\xi \in \Sigma := \{\xi_1 = 0\}$.

3. Let $M \subset \mathbb{R}^n$ be a smooth hypersurface, locally described by an equation of the form $x_n = h(x')$ near the origin, where $h \in C^\infty$ and $h(x') = O(|x'|^\gamma)$ for some $\gamma \geq 2$ as $x' \to 0$. A straightforward modification of Knapp's argument shows that an analog of (2) *can only hold if*

$$q \geq \frac{2(n - 1 + \gamma)}{n - 1}, \tag{5}$$

If M is flat at some point in the sense that all its principal curvatures vanish there (equivalently: $\gamma > 2$), then the range of exponents q which are not excluded by (5) *is strictly smaller than indicated by the Tomas-Stein condition. Moreover, if M osculates its tangent plane to infinite order at some point (let $\gamma \to \infty$), then the only allowed exponent q is the trivial exponent $q = \infty$.*

4. If M_0 is a compact subset of a hypersurface M with nonvanishing Gaussian curvature, then

$$\|\widehat{f\sigma}\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)} \leq C(n, M_0)\|f\|_{L^2(M_0)} \tag{6}$$

for any $f \in C^\infty(M_0)$. Take, for instance, the truncated paraboloid:

$$M_1 := \{(x', |x'|^2) : x' \in \mathbb{R}^{n-1}, |x'| \leq 1\},$$

which is important for the Schrödinger equation. On the other hand, (6) *fails for the piece of cone*

$$M_2 := \{(x', |x'|) : x' \in \mathbb{R}^{n-1}, 1 \leq |x'| \leq 2\},$$

since it has exactly one vanishing principal curvature. However, if $n \geq 3$ and we equip the piece of cone $M_2 \subset \mathbb{R}^n$ with the measure $d\mu(x) = \frac{dx'}{|x'|}$, we still have that[7]*:*

$$\|\widehat{f\sigma}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \lesssim \|f\|_{L^2(M_2,\mu)}.$$

The Bessel function can also be defined for real (nonintegral) values of m; when $m > -1/2$ it is given by

$$J_m(r) := \frac{(r/2)^m}{\Gamma(m + 1/2)\pi^{1/2}} \int_{-1}^{1} e^{irt}(1 - t^2)^{m-1/2} dt.$$

To check that this second definition agrees with the earlier one (when m is a positive integer), note first that this identity is evident when $m = 0$. Next, a straightforward computation shows that both expressions satisfy the recursion relation

$$\frac{d}{dr}[r^{-m} J_m(r)] = -r^{-m} J_{m+1}(r).$$

[7]Cf. [6], Theorem 16, p.77.

2. Proof of the Tomas-Stein inequality

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a Schwartz function in $\mathbb{R}^n$ and $\epsilon > 0$, then we define:

$$\varphi_\epsilon(x) := \varphi(\epsilon x) \text{ and } \varphi^\epsilon(x) := \epsilon^{-n}\varphi(\epsilon^{-1}x).$$

It follows that $\widehat{\varphi_\epsilon} = (\widehat{\varphi})^\epsilon$ and $\widehat{\varphi^\epsilon} = (\widehat{\varphi})_\epsilon$.

We now present a partial proof of Theorem 2 under the weaker assumption $q > \frac{2(n+1)}{n-1}$. The proof of the endpoint case is harder and requires complex interpolation (cf. chapter 9 of [6], or [7]). Real interpolation (in the form of Riesz-Thörin) and one of its consequences (Young's convolution inequality) is all we need in the following:

Proof. According to lemma 1, it will be enough to show that, if $q > \frac{2(n+1)}{n-1}$, then

$$\|f * \widehat{\sigma}\|_q \lesssim \|f\|_{q'}$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$. The properties of σ that will be relevant to us are only two:

(i) (decay) $|\widehat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-\frac{(n-1)}{2}}$;
(ii) (dimensionality) $\sigma(D(x, r)) \lesssim r^{n-1}$.

Start by considering a smooth function $\phi \in C_0^\infty(\mathbb{R}^n)$ supported on $\{\frac{1}{4} \leq |x| \leq 1\}$ and such that

$$\sum_{j=0}^{\infty} \phi(2^{-j}x) = 1 \text{ if } |x| \geq 1.$$

A way to do this is to take $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x) \equiv 0$ for $|x| \leq 1/2$ and $\chi(x) \equiv 1$ for $|x| \geq 1$, and let $\phi(x) := \chi(2x) - \chi(x)$.

Let:

$$\begin{aligned}\widehat{\sigma}(x) =& \Big[1 - \sum_{j=0}^{\infty} \phi(2^{-j}x)\Big]\widehat{\sigma}(x) + \sum_{j=0}^{\infty} \phi(2^{-j}x)\widehat{\sigma}(x) \\ =:& K_{-\infty}(x) + \sum_{j=0}^{\infty} K_j(x).\end{aligned}$$

Since $K_{-\infty} \in C_0^\infty(\mathbb{R}^n)$ and $q > 2$, the estimate

$$\|f * K_{-\infty}\|_q \lesssim \|f\|_{q'}$$

follows from Young. For finite j, the logic will be to interpolate between a $L^1 \to L^\infty$ bound and a $L^2 \to L^2$ bound. Indeed, we have the following estimates:

(1) $\|f * K_j\|_\infty \leq \|K_j\|_\infty \|f\|_1 \lesssim 2^{-j\frac{n-1}{2}} \|f\|_1$;
(2) $\|f * K_j\|_2 = \|\widehat{f}\widehat{K_j}\|_2 \leq \|\widehat{K_j}\|_\infty \|f\|_2 \lesssim 2^j \|f\|_2$.

The first estimate (1) follows from property (i) of σ and the fact that $\phi_{2^{-j}}$ is localized to $x \sim 2^j$:

$$|K_j(x)| = |\phi(2^{-j}x)\widehat{\sigma}(x)| \lesssim (1 + 2^j)^{-\frac{n-1}{2}} \leq 2^{-j\frac{n-1}{2}}.$$

The second estimate (2) is more delicate. This is where we make use of (ii). Letting $\psi = \widehat{\phi}$, we have that

$$\widehat{K_j}(\xi) = (\psi^{2^{-j}} * \sigma)(\xi) = 2^{jn} \int_{S^{n-1}} \psi(2^j(\xi - \eta)) d\sigma(\eta).$$

Since $\psi \in \mathcal{S}(\mathbb{R}^n)$, it follows that:

$$|\widehat{K_j}(\xi)| \le C_N 2^{jn} \int (1 + 2^j|\xi - \eta|)^{-N} d\sigma(\eta), \text{ for every } N \in \mathbb{N}.$$

Taking $N = n$, we have that:

$$\begin{aligned}
|\widehat{K_j}(\xi)| \le & C_n 2^{jn} \int (1 + 2^j|\xi - \eta|)^{-n} d\sigma(\eta) \\
= & C_n 2^{jn} \Big[\int_{D(\xi,2^{-j})} (1 + 2^j|\xi - \eta|)^{-n} d\sigma(\eta) + \sum_{k=1}^{\infty} \int_{D(\xi,2^{-j+k})\setminus D(\xi,2^{-j+k-1})} (1 + 2^j|\xi - \eta|)^{-n} d\sigma(\eta) \Big] \\
\lesssim & 2^{jn} \Big[\sigma(D(\xi, 2^{-j})) + \sum_{k=1}^{\infty} \int_{D(\xi,2^{-j+k})\setminus D(\xi,2^{-j+k-1})} (2^{k-1})^{-n} d\sigma(\eta) \Big] \\
\lesssim & 2^{jn} \Big[(2^{-j})^{n-1} + \sum_{k=1}^{\infty} 2^{-n(k-1)} \sigma(D(\xi, 2^{-j+k}) \setminus D(\xi, 2^{-j+k-1})) \Big] \\
\lesssim & 2^{jn} \Big[2^{-j(n-1)} + \sum_{k=1}^{\infty} 2^{-n(k-1)} (2^{-j+k-1})^{n-1} \Big] \\
\lesssim & 2^j.
\end{aligned}$$

Interpolating between (1) and (2), we get that:

$$\|f * K_j\|_q \lesssim 2^{j\theta} 2^{-j\frac{n-1}{2}(1-\theta)} \|f\|_{q'}$$

if $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{\infty}$. This is equivalent to

$$\|f * K_j\|_q \lesssim (2^{-j})^{-\frac{2}{q}+\frac{n-1}{2}(1-\frac{2}{q})} \|f\|_{q'}.$$

We conclude the proof by noting that $-\frac{2}{q} + \frac{n-1}{2}\Big(1 - \frac{2}{q}\Big) > 0$ iff $q > \frac{2(n+1)}{n-1}$ and invoking Fatou's lemma.

□

Some remarks about the proof:

Remark 7. *No special properties of the sphere were used in this proof. All we needed was conditions (i) and (ii), and that holds whenever we are dealing with a compact submanifold of codimension 1 and nonvanishing Gaussian curvature.*

Remark 8. *The best constant in* (2) *is unknown, even in the case of* $S^2 \subset \mathbb{R}^3$*. Keeping track of the constants in the proof above, we have that*

$$\|f * \widehat{\sigma}\|_q \leq C_n \|f\|_{q'}$$

where $C_n = (3A_n C_n 2^{n-1})^{2/q}$*, where* A_n *and* C_n *are the constants such that* $\sigma(D(x,r)) \leq A_n r^{n-1}$ *and* $|\widehat{\sigma}(\xi)| \leq C_n(1+|\xi|)^{-n}$*. That this is not the optimal constant should be anticipated from the fact that we did not use the sharp form of Young's convolution inequality: letting* $C_p := \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{1/2}$*, we have that*

$$\|f * g\|_r \leq \left(\frac{C_p C_q}{C_r}\right)^n \|f\|_p \|g\|_q$$

whenever $p, q, r \geq 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$*, and* $f \in L^p(\mathbb{R}^n)$*,* $g \in L^q(\mathbb{R}^n)$ *are nonnegative functions.*

Specialize to $n = 3$*. Then the Tomas-Stein inequality for the sphere reads:*

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \lesssim \|f\|_{L^2(S^2)}.$$

We will see in the next section that, even though the proof carries through for the paraboloid $\mathbb{P}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = \frac{1}{2}(x_1^2 + x_2^2)\}$ *(note that* $\mathbb{P}^2$ *has the same Gaussian curvature as* S^2 *and its surface measure* σ_P *also satisfies (i) and (ii)), the optimal constants in the corresponding inequalities are different. Letting*

$$\mathcal{R}_{S^2} := \sup_{0 \neq f \in L^2(S^2)} \frac{\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)}}{\|f\|_{L^2(S^2,\sigma)}}$$

and

$$\mathcal{R}_{\mathbb{P}^2} := \sup_{0 \neq g \in L^2(\mathbb{P}^2)} \frac{\|\widehat{g\sigma}\|_{L^4(\mathbb{R}^3)}}{\|g\|_{L^2(\mathbb{P}^2,\sigma_P)}}$$

we will show that

$$\mathcal{R}_{S^2} > \mathcal{R}_{\mathbb{P}^2}. \tag{7}$$

Whereas the value of the constant $\mathcal{R}_{\mathbb{P}^2}$ *is known* [4][5]*, the determination of* $\mathcal{R}_{S^2}$ *remains open.*

Remark 9. *An application of the Tomas-Stein inequality is the derivation of Strichartz estimates for partial differential equations. These bound the (total)* L^p *norm of the solution in terms of Banach space norms of the initial data. A simple example is the homogeneous Schrödinger equation in* $\mathbb{R}^n$

$$-i\partial_t u + \frac{1}{2\pi}\Delta u = 0$$

with initial condition $u(x,0) = f(x)$*. If* μ *is the measure in* $\mathbb{R}^n \times \mathbb{R}$ *given by*

$$\int_{\mathbb{R}^n \times \mathbb{R}} \phi(\xi, t) d\mu = \int_{\mathbb{R}^n} \phi(\xi, |\xi|^2) d\xi,$$

then the Tomas-Stein inequality (for the paraboloid) implies

$$\|u\|_{L^p(\mathbb{R}^{n+1})} = \|(\widehat{f}\mu)\check{}\|_{L^p(\mathbb{R}^{n+1})} \lesssim \|\widehat{f}\|_{L^2(\mu)} = \|f\|_{L^2(\mathbb{R}^n)}$$

if $p \geq 2 + \frac{4}{n}$.

3. On the existence of extremizers ($n = 3$)

If $n = 3$, then the Tomas-Stein exponent $\frac{2(n+1)}{n-1} = 4$ is an even integer. When studying the inequality

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leq C\|f\|_{L^2(S^2)}, \tag{8}$$

we can make use of bilinear convolution estimates, a major advantage which is unavailable in higher dimensions (still OK if $n = 2$ though). On the one hand[8] ,

$$\|\widehat{f\sigma}\|_4 = \|\widehat{f\sigma}\widehat{f\sigma}\|_2^{1/2} = \|\widehat{f\sigma * f\sigma}\|_2^{1/2} = (2\pi)^{3/4}\|f\sigma * f\sigma\|_2^{1/2}$$

by Plancherel. On the other hand, this together with Hölder and Tomas-Stein imply that

$$\|f\sigma * g\sigma\|_2 = \|\widehat{f\sigma}\widehat{g\sigma}\|_2 \leq \|\widehat{f\sigma}\|_4\|\widehat{g\sigma}\|_4 \lesssim \|f\|_{L^2(S^2)}\|g\|_{L^2(S^2)}.$$

Definition 10. *An* **extremizing sequence** *for the inequality* (8) *is a sequence* $\{f_n\}$ *of functions in* $L^2(S^2)$ *satisfying* $\|f_n\|_{L^2(S^2)} \leq 1$ *such that* $\|\widehat{f_n\sigma}\|_{L^4(\mathbb{R}^3)} \to \mathcal{R}_{S^2}$ *as* $n \to \infty$. *An* **extremizer** *for the inequality* (8) *is a function* $f \neq 0$ *which satisfies* $\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} = \mathcal{R}_{S^2}\|f\|_{L^2(S^2)}$.

In the remainder of this paper, we address the main result of [2]:

Theorem 11. *There exists an extremizer in* $L^2(S^2)$ *for the inequality* (8). *In fact, any extremizing sequence in* $L^2(S^2)$ *of nonnegative functions for the inequality* (8) *is precompact.*

Remark 12. *Recall that a sequence of functions in* $L^2(S^2)$ *is precompact if any subsequence has a sub-subsequence which is Cauchy in* $L^2(S^2)$. *The first assertion follows from the second since* L^2 *limits of extremizing sequences are extremizers. Finally, nonnegative functions play a special role because of the pointwise inequality:*

$$\|\widehat{f\sigma}\|_4 = \|f\sigma * f\sigma\|_2^{1/2} \leq \||f|\sigma * |f|\sigma\|_2^{1/2} = \|\widehat{|f|\sigma}\|_4.$$

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[8]Recall the definition of the convolution of two (integrable) functions supported on the sphere:

$$(f\sigma * g\sigma)(x) = \frac{\chi_{B(0,2)}(x)}{|x|}\int_0^{2\pi} f(h_x(\theta))g(h_x(\theta+\pi))d\theta,$$

where $h_x(\theta) = \frac{x}{2} + r_x(u_x\cos\theta + v_x\sin\theta)$ and $r_x = (1 - \frac{|x|^2}{4})^{1/2}$.

3.1. **The sphere and the paraboloid.** Our first goal is to establish the inequality (7) advertised in remark 8. We will work with the (sharp) bilinear inequalities:

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leq \mathbf{S}^2 \|f\|_{L^2(S^2,\sigma)}; \tag{9}$$

$$\|g\sigma_P * g\sigma_P\|_{L^2(\mathbb{R}^3)} \leq \mathbf{P}^2 \|g\|_{L^2(\mathbb{P}^2,\sigma_P)}. \tag{10}$$

As we remarked before, the inequality for the paraboloid is related to Strichartz estimates for the Schrödinger equation. In [4],[5] it is proved that any Gaussian is an extremizer for the paraboloid, and conversely. More precisely, we have the following theorem:

Theorem 13. *(Gaussian extremizers: Theorem 1.5., [5]) The function $f \in L^2(\mathbb{R}^2)$ is an extremizer for the sharp Strichartz inequality*

$$\|e^{it\Delta} f\|_{L^4(\mathbb{R}\times\mathbb{R}^2)} \leq \mathcal{R}\|f\|_{L^2(\mathbb{R}^2)}$$

if and only if f is a Gaussian, that is,

$$f(x) = Ae^{(\lambda+i\mu)|x-a|^2+b\cdot x},$$

where $A \in \mathbb{C}$, $\lambda > 0$, $\mu \in \mathbb{R}$,$a \in \mathbb{R}^2$ and $b \in \mathbb{C}^2$.

Using this, we can prove the following lemma:

Lemma 14. $\mathbf{S} \geq (\frac{3}{2})^{1/4}\mathbf{P}$.

Proof. Let $\epsilon > 0$. By Theorem 13, we can choose $0 \leq g \in L^2(S^2)$ supported in a sufficiently small open disk and satisfying $\|g\sigma * g\sigma\|_2 \geq (\mathbf{P} - \epsilon)^2 \|g\|^2_{L^2(S^2)}$. Set $d\mu = gd\sigma$, $f = g + \widetilde{g}$ and $d\nu = fd\sigma = d\mu + d\widetilde{\mu}$. Since g and $\widetilde{g}$ have disjoint supports,

$$\|f\|^2_{L^2(S^2)} = 2\|g\|^2_{L^2(S^2)}.$$

On the other hand,

$$\nu * \nu = (\mu + \widetilde{\mu}) * (\mu + \widetilde{\mu}) = \mu * \mu + \widetilde{\mu} * \widetilde{\mu} + 2(\mu * \widetilde{\mu}).$$

Again by disjointness of supports,

$$\begin{aligned} \|\nu * \nu\|^2_{L^2(\mathbb{R}^3)} \geq& \|\mu * \mu\|^2_{L^2} + \|\widetilde{\mu} * \widetilde{\mu}\|^2_{L^2} + 4\|\mu * \widetilde{\mu}\|^2_{L^2} \\ =& 6\|\mu * \mu\|^2_{L^2}. \end{aligned}$$

Since $\|f\|^2_{L^2(S^2)} = 2\|g\|^2_{L^2(S^2)}$, we obtain a ratio $\frac{6}{4} = \frac{3}{2}$. Thus $\mathbf{S} \geq (\frac{3}{2})^{1/4}\mathbf{P}$. □

Remark 15. *Working a bit harder we can prove more, namely $\mathbf{S} \geq 2^{1/4}\mathbf{P}$. This is done by computing $\frac{\|1\sigma*1\sigma\|_2^2}{\|1\|^4_{L^2(\sigma)}} = \frac{8\pi a^2}{(4\pi)^2}$ and $\frac{\|F\sigma_P*F\sigma_P\|_2^2}{\|F\|^4_{L^2(\sigma_P)}} = \frac{\pi a^2/4}{\pi^2}$ for $F(x', x_3) = e^{-|x'|^2/2} \equiv e^{-x_3}$ (which is an extremizer by Theorem 13).*

The proof shows that the fundamental potential obstruction to the precompactness of (nonnegative) extremizing sequences is the possibility that f_n^2 could converge weakly to a Dirac mass (and so $f_n \rightharpoonup 0$), or to a linear combination of two Dirac masses at a pair of antipodal points (see corollary 19 below). Lemma 14 and the result mentioned in remark 15 are essential in proving that this cannot happen.

3.2. **A technical lemma.** Choose a "gauge" function $\Theta : [1,\infty) \to (0,\infty)$ satisfying $\Theta(R) \to 0$ as $R \to \infty$. Let a sequence of functions $\{g_n\} \subset L^2(\mathbb{R}^2)$ satisfy:

$$g_n \geq 0, \tag{11}$$

$$\|g_n\|_2 \to 1, \tag{12}$$

$$\int_{|x|\geq R} g_n(x)^2 dx \leq \Theta(R), \tag{13}$$

$$\int_{g_n(x)\geq R} g_n(x)^2 dx \leq \Theta(R). \tag{14}$$

The following lemma is the only place in the analysis of [2] where the nonnegativity of an extremizing sequence is used:

Lemma 16. *If* $\{g_n\}$ *satisfies hypothesis* (11)-(14) *above, then for any* $A > 0$ *there exists* $c > 0$ *such that*

$$\int_{|\xi|\leq A} |\widehat{g_n}(\xi)|^2 d\xi \geq c$$

uniformly in n.

Proof. Let $g \in L^2(\mathbb{R}^2)$ be a nonnegative function which satisfies $\|g\|_2 = 1$ and the inequalities (13), (14). For $t > 0$, let $\varphi_t(y) = e^{-t|y|^2/2}$. Then $\widehat{\varphi}_t(\xi) = 2\pi t^{-1} e^{-|\xi|^2/2t}$ and

$$\int g\varphi_t dy = (2\pi)^{-2} \int \widehat{g}(\xi)\widehat{\varphi}_t(\xi) d\xi = (2\pi)^{-1} t^{-1} \int \widehat{g}(\xi) e^{-|\xi|^2/2t} d\xi.$$

For any $R, \rho \geq 1$, let

$$S := \{y \in \mathbb{R}^2 : |y| \leq R \text{ and } g(y) \leq \rho\}.$$

Choose R, ρ large enough so that $\Theta(R) + \Theta(\rho) \leq \frac{1}{2}$. Then, for any $t > 0$:

$$\begin{aligned}
\int_{\mathbb{R}^2} g\varphi_t dy \geq & e^{-tR^2/2} \int_S g(y)dy \\
\geq & e^{-tR^2/2} \rho^{-1} \int_S g^2(y)dy \\
= & e^{-tR^2/2} \rho^{-1} \Big(\|g\|_2^2 - \int_{\mathbb{R}^2 \setminus S} g^2(y)dy \Big) \\
\geq & \frac{1}{2} e^{-tR^2/2} \rho^{-1}.
\end{aligned}$$

On the other hand, by Cauchy-Schwarz:

$$\int_{|\xi|\geq A}|\widehat{g}(\xi)|t^{-1}e^{-|\xi|^2/2t}d\xi \leq Ct^{-1}\|\widehat{g}\|_2\Big(\int_{|\xi|\geq A}e^{-|\xi|^2/t}d\xi\Big)^{1/2}$$
$$=Ct^{-1}\Big(\int_{r=A}^{\infty}e^{-r^2/t}2rdr\Big)^{1/2}=Ct^{-1/2}e^{-A^2/2t}.$$

Cauchy-Schwarz also gives

$$\int_{|\xi|\leq A}|\widehat{g}(\xi)|t^{-1}e^{-|\xi|^2/2t}d\xi \leq Ct^{-1/2}\Big(\int_{|\xi|\leq A}|\widehat{g}(\xi)|^2d\xi\Big)^{1/2}.$$

It follows that

$$Ct^{-1/2}\Big(\int_{|\xi|\leq A}|\widehat{g}(\xi)|^2d\xi\Big)^{1/2} \geq \int_{\mathbb{R}^2}|\widehat{g}(\xi)|t^{-1}e^{-|\xi|^2/2t}d\xi - \int_{|\xi|\geq A}|\widehat{g}(\xi)|t^{-1}e^{-|\xi|^2/2t}d\xi$$
$$\geq Ce^{-tR^2/2}\rho^{-1} - Ct^{-1/2}e^{-A^2/2t}.$$

Substitute $t = A^2/\gamma$ where $\gamma = \gamma(A) \geq 1$ to get

$$C\gamma^{1/2}A^{-1}\Big(\int_{|\xi|\leq A}|\widehat{g}(\xi)|^2d\xi\Big)^{1/2} \geq Ce^{-A^2R^2/2\gamma}\rho^{-1} - C\gamma^{1/2}A^{-1}e^{-\gamma/2}.$$

R, ρ have been fixed independently of A. As all these three quantities remain fixed and $\gamma \to \infty$, this last lower bound tends to $\pi\rho^{-1} - 0 > 0$. Thus choosing γ sufficiently large yields the desired bound. □

3.3. **Idea of the Proof.** We say that $f \in L^2(S^2)$ is **upper normalized** with respect to Θ and a cap $\mathcal{C}(z, r)$ if (13) and (14) hold for $|x - z| \geq Rr$ and $|f(x)| \geq Rr^{-1}$, respectively. An even function is said to be **upper even-normalized** wrt $\Theta, \mathcal{C}$ if $f = f_+ + f_-$ where $f_-(x) \equiv \overline{f_+(-x)}$ and f_+ is upper normalized wrt $\Theta, \mathcal{C}$.

Proposition 17. *There exists a gauge function $\Theta : [1,\infty) \to (0,\infty)$ satisfying $\Theta(R) \to 0$ as $R \to \infty$ with the following property. For any $\epsilon > 0$ there exists $\delta > 0$ such that any nonnegative even function $f \in L^2(S^2)$ satisfying $\|f\|_{L^2(S^2)} = 1$ which is δ-nearly extremal*[9] *may be decomposed as $f = G + H$ where G, H are even and nonnegative with disjoint supports, $\|H\|_2 < \epsilon$ (small!) and there exists a cap $\mathcal{C}$ such that G is upper even-normalized with respect to $\mathcal{C}, \Theta$ (structured!).*

Proposition 18. *Let $\{f_n\} \subset L^2(S^2)$ be an extremizing sequence of nonnegative even functions for the inequality (9), satisfying $\|f_n\|_{L^2(S^2)} = 1$. Suppose that each f_n is upper even-normalized with respect to a cap $\mathcal{C}_n = \mathcal{C}(z_n, r_n)$, with constants uniform in n. Then for any $\epsilon > 0$ there exists $C_\epsilon < \infty$ with the following property. For every n, if $r_n \leq \frac{1}{2}$,*

[9] A nonzero function $f \in L^2(S^2)$ is said to be **δ-nearly extremal** for the inequality (9) if

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \geq (1-\delta)^2\mathbf{S}^2\|f\|_2^2.$$

then[10] $\phi^*_{\mathcal{C}_n}(f_n) =: \phi^*_n(f_n)$ *may be decomposed as* $\phi^*_n(f_n) = G_n + H_n$*, where* H_n *is small and* G_n *is structured in the sense that:*

$$\|H_n\|_2 < \epsilon, \tag{15}$$

$$supp(G_n) \subset \{|x| \le C_\epsilon\}, \tag{16}$$

$$\|G_n\|_{C^1} \le C_\epsilon. \tag{17}$$

If $r_n \ge \frac{1}{2}$*, then* f_n *itself may be decomposed as* $f_n = g_n + h_n$ *where* $\|h_n\|_2 < \epsilon$ *and* $\|g_n\|_{C^1} \le C_\epsilon$*.*

The rough idea of the proof of Lemma 18 is the following: from lemma 16 we know that if $g \in L^2(\mathbb{R}^2)$ satisfies $\|g\|_2 \sim 1$, if g is upper normalized with respect to the unit ball, and if g is nonnegative, then $\int_{|\xi|\lesssim 1} |\widehat{g}(\xi)|^2 d\xi$ is bounded below by a universal strictly positive constant. If conclusion (17) of the lemma failed, then $g_n = \phi^*_n(f_n)$ would have to satisfy $\int_{|\xi|\ge\Lambda_n} |\widehat{g_n}(\xi)|^2 d\xi \ge \eta > 0$, with $\limsup_n \Lambda_n = \infty$. Thus in an appropriately rescaled sense f_n is a superposition of a slowly varying part, plus a highly oscillatory part, with perhaps some intermediate portion. This is a classical situation in the theory of concentration compactness. For the bilinear expression $f_n\sigma * f_n\sigma$, the cross term resulting from the high and the low frequency parts is shown to be small (details omitted), and this contradicts extremality.

An application of the Rellich-Kondrachov compactness theorem (which states, in particular, that for any $\epsilon > 0$ and bounded open subset $U \subset \mathbb{R}^2$ with a C^1 boundary, $W^{1,1+\epsilon}(U) \subset\subset L^2(U)$) then yields:

Corollary 19. *Let* $\{f_n\} \subset L^2(S^2)$ *be an extremizing sequence of even nonnegative functions for the inequality* (9)*, which are upper even-normalized with respect to a sequence of caps* $\{\mathcal{C}(z_n, r_n)\}$*.*

(i) If $r_n \to 0$*, then* $\{\phi^*_n(f_n)\}$ *is precompact in* $L^2(\mathbb{R}^2)$*;*

(ii) If $\liminf_n r_n > 0$*, then* $\{f_n\}$ *is precompact in* $L^2(S^2)$*.*

One finally proves that, for a sequence $\{f_n\}$ as in proposition 18, $\liminf_n r_n > 0$ (this uses Lemma 14 and prevents $f_n^2 \rightharpoonup \delta_z, \frac{\delta_z+\delta_{-z}}{2}$, as previously mentioned), and this concludes the analysis.

3.4. **Open questions.** The following are unsolved problems:

1. Do constant functions extremize the inequality (8)?
2. Are extremizers unique modulo rotations (and, in case of complex-valued functions, multiplication by $ce^{ix\cdot\xi}$)?
3. What is the value of $\mathbf{S}$?
4. Existence of extremizers for $n > 3$.
5. The case of compact perturbations of $S^2 \subset \mathbb{R}^3$.

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[10] Recall the definition of the pullbacks: any cap of radius ≤ 1 is associated to a rescaling map $\phi_{\mathcal{C}} : B(0,1) \to \mathcal{C}$. Then the pullback $\phi^*_{\mathcal{C}} f(y) := r(f \circ \phi_{\mathcal{C}})(y)$ is such that $\|\phi^*_{\mathcal{C}} f\|_{L^2(\mathbb{R}^2)} \asymp \|f\|_{L^2(S^2)}$ with the ratio of these norms being bounded above and below by positive, finite constants, uniformly in f, r, z.

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