1 On multilinear oscillatory integrals, nonsingular and singular

after M. Christ, X. Li, T. Tao and C. Thiele [2]
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Abstract

We explore the relationship between decay estimates for certain multilinear oscillatory integrals and nondegeneracy of the corresponding polynomial phase.

1.1 Introduction

The classical theory of oscillatory integrals of the second kind [3] establishes the boundedness of operators of the form

$$(T_\lambda f)(\xi) = \int_{\mathbb{R}^m} e^{i\lambda\Phi(x,\xi)} f(x)\psi(x,\xi)dx.$$

Here ψ is a fixed smooth function of compact support in x and ξ, the phase Φ is real-valued and smooth and the Hessian of Φ is nonvanishing on the support of ψ. In this case,

$$\|T_\lambda(f)\|_{L^2(\mathbb{R}^m)} \leq C\lambda^{-m/2}\|f\|_{L^2(\mathbb{R}^m)}.$$

Aiming at similar results in a somewhat different context, we start by considering multilinear oscillatory operators of the form

$$\Lambda_\lambda(f_1, \dots, f_n) = \int_{\mathbb{R}^m} e^{i\lambda P(x)} \prod_{j=1}^{n} f_j \circ \pi_j(x)\eta(x)dx.$$

Here $P : \mathbb{R}^m \to \mathbb{R}$ is a real-valued polynomial, $\pi_j : \mathbb{R}^m \to V_j$ are orthogonal projections onto some subspaces V_j of $\mathbb{R}^m$, $f_j : V_j \to \mathbb{C}$ are locally integrable functions with respect to Lebesgue measure on V_j, and $\eta \in C_0^1(\mathbb{R}^m)$ is compactly supported. We assume that all the subspaces V_j have the same dimension, which we denote by κ.

Definition 1. *A polynomial P has the* power decay property with respect to $\{V_j\}_{j=1}^n$ in an open set $U \subset \mathbb{R}^m$ *if for any $\eta \in C_0^1(U)$, there exist $\epsilon > 0$ and $C < \infty$ such that*

$$|\Lambda_\lambda(f_1, \ldots, f_n)| \leq C(1 + |\lambda|)^{-\epsilon} \prod_{j=1}^n \|f_j\|_{L^\infty(V_j)}, \quad \forall f_j \in L^\infty(V_j), \forall \lambda \in \mathbb{R}. \quad (1)$$

The goal is to characterize data $(P, \{V_j\}_j)$ for which the power decay property holds, and this is accomplished to a significant but incomplete extent in [2].

Observe that, if $P = \sum_j f_j \circ \pi_j$ for some measurable functions f_j, then (1) cannot hold. This motivates the following definition:

Definition 2. *A polynomial P is* degenerate with respect to $\{V_j\}_{j=1}^n$ *if $P = \sum_{j=1}^n p_j \circ \pi_j$ for some polynomials $p_j : V_j \to \mathbb{R}$. Otherwise P is said to be* nondegenerate*. If $n = 0$, P is degenerate if and only if P is constant.*

An important definition is not complete without a good example:

Example 3. *In $\mathbb{R}^3$, let $P(x) = x_3^2$ and $L = \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$. Let $n \geq 1$ be arbitrary. For $1 \leq j \leq n$, take light-cone unit vectors $v_j = (v_j^1, v_j^2, v_j^3) \in \mathbb{R}^3$ such that $(v_j^3)^2 = (v_j^1)^2 + (v_j^2)^2$. Let π_j be the orthogonal projection onto $V_j := \text{span }(v_j)$, that is, $\pi_j(x) = x \cdot v_j = x_1 v_j^1 + x_2 v_j^2 + x_3 v_j^3$. One readily checks that P is nondegenerate with respect to $\{V_j\}_{j=1}^n$, for every $n \in \mathbb{N}$. This is surprising in view of the following fact: in $\mathbb{R}^2$, any polynomial $Q : \mathbb{R}^2 \to \mathbb{R}$ of degree two is degenerate with respect to any family of three or more mappings of the form $\pi_j(x) = x \cdot w_j$ (where none of the w_j is a multiple of any of the others).*

The nondegeneracy condition is to replace the hypothesis of a nonvanishing derivative in the result about oscillatory integrals of the second kind. The question is then the following:

Question 4. *Is the power decay property equivalent to nondegeneracy?*

The cases $n = 0, 1$ fall into the theory of oscillatory integrals of the first kind. In the case $n = 2$ one is dealing with bilinear forms $\langle T_\lambda(f_1), f_2 \rangle$, where the associated operators T_λ are of the form discussed above. If $n \geq 3$ and $m < n\kappa$, however, there arises a class of singular oscillatory integrals which have no direct analogues in the bilinear case.

1.2 Algebraic aspects of nondegeneracy

We begin our discussion by introducing a new definition:

Definition 5. *A polynomial P is* simply nondegenerate with respect to $\{V_j\}_{j=1}^n$ *if there exists a differential operator of the form $L = \prod_{j=1}^n w_j \cdot \nabla$ with $w_j \in V_j^\perp$ and such that $L(P)$ does not vanish identically.*

Simple nondegeneracy implies nondegeneracy, but the converse does not hold in general (example 3). The converse does hold, however, in the following two special cases:

(i) If $\kappa = m - 1$;

(ii) If $n(m - \kappa) \leq m$ and the V_j's are in general position[1].

We have the following characterization of nondegeneracy:

Lemma 6. *Let P be a real-valued polynomial of degree d. Then:*

(i) P is nondegenerate if and only if there exists a constant-coefficient partial differential operator L such that $L(P) \neq 0$ but $L(\sum_j p_j \circ \pi_j) = 0$ for every polynomial p_j of degree d;

(ii) P is degenerate if and only if $P = \sum_j h_j \circ \pi_j$ for some distributions $h \in \mathcal{D}'(V_j)$;

(iii) P is nondegenerate if and only if one of its homogeneous summands is nondegerate.

In the case of homogeneous polynomials, we can refine our characterization as follows:

Lemma 7. *Let P be a homogeneous polynomial of degree d. Then:*

(i') P is nondegenerate if and only if there exists a constant-coefficient partial differential operator L, homogeneous of degree d, such that $L(P) \neq 0$ but $L(\sum_j p_j \circ \pi_j) = 0$ for every polynomial p_j of degree d;

(ii') P is degenerate if and only if $P = \sum_j p_j \circ \pi_j$ for some homogeneous polynomials p_j of degree d.

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[1] In this context, a family of subspaces $\{V_j\}_{j=1}^n$ of $\mathbb{R}^m$ of dimension κ is said to be in *general position* if any subfamily of cardinality $k \geq 1$ spans a subspace of dimension $\min\{k\kappa, m\}$.

1.3 Main results

1.3.1 Further definitions

Let $\mathcal{P}_{\leq d}$ denote the finite-dimensional vector space of polynomials in $\mathbb{R}^m$ of degree $\leq d$, endowed with a metric $\|\cdot\|$. Given d, the norm of P with respect to $\{V_j\}_{j=1}^n$ is defined to be

$$[P]_{d,\{V_j\}_j} := \inf_{\deg p_j \leq d} \|P - \sum_{j=1}^n p_j \circ \pi_j\|.$$

This indeed defines a norm on the quotient space $\mathcal{P}_{\leq d}$ modulo degenerate polynomials.

Definition 8. *A family of polynomials* $\{P_\alpha\}_\alpha$ *is* uniformly nondegenerate with respect to $\{V_j\}_{j=1}^n$ *if there exist* $d < \infty$ *and* $c > 0$ *such that*

$$\sup_\alpha \deg P_\alpha \leq d \quad \textit{and} \quad \inf_\alpha [P_\alpha]_{d,\{V_j\}_j} \geq c.$$

Definition 9. *The collection* $\{V_j\}_{j=1}^n$ *has the* power decay property *if every polynomial* P *which is nondegenerate with respect to* $\{V_j\}_j$ *has the power decay property* (1) *in every open set* U*. The power decay is* uniform *if* (1) *holds with uniform constants* C, ϵ *for any family of polynomials which are uniformly nondegenerate with respect to* $\{V_j\}_j$*.*

1.3.2 Decay for nonsingular multilinear oscillatory integrals

The first result states that a simply nondegenerate polynomial has the power decay property in every open set. More precisely:

Theorem 10. *Fix* $d \in \mathbb{N}$ *and* $c > 0$*. Consider the operator* $L = \prod_j w_j \cdot \nabla$ *with* $w_j \in V_j^\perp$ *and* $\|w_j\| = 1$*. Then there exist* $C < \infty$ *and* $\epsilon > 0$ *with the following property: if* P *is a polynomial such that* $\deg P \leq d$ *and* $\max_{|x|\leq 1} |L(P)(x)| \geq c$*, then* (1) *holds.*

As corollaries, we get that families of codimension one subspaces have the uniform power decay property, as do families of small[2] codimension subspaces in general position.

The second result tells us that the same conclusion still holds in the one-dimensional case, provided we do not consider "too many" subspaces:

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[2] Here, "small" means that $n(m - \kappa) \leq m$.

Theorem 11. *If $n < 2m$, then any family $\{V_j\}_j$ of one-dimensional subspaces which lie in general position has the uniform power decay property. Moreover under these assumptions*

$$|\Lambda_\lambda(f_1,\dots,f_n)| \le C(1+|\lambda|)^{-\epsilon}\prod_{j=1}^{n}\|f_j\|_{L^2(\mathbb{R})}, \quad \forall f_j \in L^2(\mathbb{R}), \forall\lambda\in\mathbb{R} \tag{2}$$

uniformly for all polynomials P which are uniformly nondegenerate with respect to $\{V_j\}_j$.

The rest of this paper is devoted to outlining the proofs of theorem 10 (the simply nondegenerate case) and theorem 11 (the case $\kappa = 1$). Both proofs are by induction on the number of subspaces, the base case $n = 0$ being a straightforward consequence of the well-known theory of oscillatory integrals of the first kind.

1.4 The simply nondegenerate case

In this section we sketch the proof of the fact that families of codimension one subspaces have the uniform power decay property. This turns out to be equivalent to theorem 10.

We start by expressing $\Lambda_\lambda(f_1,\dots,f_n) = \langle T_\lambda(f_1,\dots,f_{n-1}), \bar{f}_n\rangle$. By Cauchy-Schwarz, it is enough to show the existence of $C < \infty$ and $\epsilon > 0$ such that

$$\|T_\lambda(f_1,\dots,f_{n-1})\|_2 \le C\lambda^{-\epsilon}\prod_{j=1}^{n-1}\|f_j\|_\infty \text{ for } |\lambda|\ge 1.$$

Choose coordinates $x = (y,z) \in \mathbb{R}^{m-1}\times\mathbb{R}$ in such a way that $V_n = \{z = 0\}$, and define $Q_\zeta(y,z) := P(y,z) - P(y,z+\zeta)$, $F_j^\zeta(\pi_j(y,z)) := f_j(\pi_j(y,z))\bar{f}_j(\pi_j(y,z+\zeta))$ and $\tilde{\eta}_\zeta(y,z) := \eta(y,z)\bar{\eta}(y,z+\zeta)$. Then

$$\begin{aligned}\|T_\lambda(f_1,\dots,f_{n-1})\|_2^2 &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}^m} e^{i\lambda Q_\zeta(x)}\prod_{j=1}^{n-1}F_j^\zeta(\pi_j(x))\tilde{\eta}_\zeta(x)dx\Big)d\zeta \\ &=: \int_{\mathbb{R}}\Lambda_\lambda^\zeta(F_1^\zeta,\dots,F_{n-1}^\zeta)d\zeta.\end{aligned}$$

Since P is nondegenerate and $\kappa = m-1$, P is simply nondegenerate. Let $L = \prod_{j=1}^n w_j\cdot\nabla$ be such that $\max_{|x|\le 1}|L(P)(x)| \ge c > 0$ and $L' := \prod_{j<n} w_j\cdot\nabla$. The key idea is to define the sublevel set

$$E_\rho := \{\zeta \in \mathbb{R} : \max_{|x|\leq 1} |L' Q_\zeta(x)| \leq \rho\}$$

and to prove the bound $|E_\rho| \leq C\rho^\delta$ for some $C < \infty$ and $\delta > 0$. For this purpose, note that the hypothesis on L and the fact that $w_n \cdot \nabla = \partial_{x_m}$ together imply that $\sup_{(x,\zeta)} |\partial_\zeta (L' Q_\zeta(x))| \geq c$. The desired bound follows from a well-known sublevel set estimate [1], provided x, ζ are restricted to lie in a fixed bounded set.

If $\zeta \notin E_\rho$, we use the induction hypothesis to conclude that

$$|\Lambda_\lambda^\zeta(F_1^\zeta, \dots, F_{n-1}^\zeta)| \leq C(1 + |\lambda|\rho)^{-\epsilon'} \prod_{j<n} \|f_j\|_\infty^2.$$

Putting everything together, we have that

$$\begin{aligned}\|T_\lambda(f_1, \dots, f_{n-1})\|_2^2 = &\int_{\mathbb{R}\setminus E_\rho} \Lambda_\lambda^\zeta(F_1^\zeta, \dots, F_{n-1}^\zeta) d\zeta + \int_{E_\rho} \Lambda_\lambda^\zeta(F_1^\zeta, \dots, F_{n-1}^\zeta) d\zeta \\ \leq &C(1 + |\lambda|\rho)^{-\epsilon'} \prod_{j<n} \|f_j\|_\infty^2 + C|E_\rho| \prod_{j<n} \|F_j^\zeta\|_\infty \\ \leq &C((|\lambda|\rho)^{-\epsilon'} + \rho^\delta) \prod_{j=1}^{n-1} \|f_j\|_\infty^2.\end{aligned}$$

Choosing $\rho = |\lambda|^{-\frac{\epsilon'}{\epsilon'+\delta}}$ yields the desired bound.

1.5 The case $\kappa = 1$

This section is devoted to presenting the main ideas behind the proof of the statement that, for a family $\{V_j\}_j$ of one-dimensional subspaces which lie in general position, the estimate (2) holds provided $n < 2m$. This last condition turns out to be necessary as well.

Since the case $n < m$ follows by a simple argument from the case $n = m$ and the theorem is already known in a more precise form when $n = m$ [3], we may assume without loss of generality that $m < n < 2m$.

Letting $A(\lambda)$ denote the best constant for which

$$|\Lambda_\lambda(f_1, \dots, f_n)| \leq A(\lambda) \prod_j \|f_j\|_2,$$

it is enough to show that $|A(\lambda)| \le C|\lambda|^{-\epsilon}$ for some $\epsilon > 0$.

In what follows we may assume that f_1 is λ-uniform in the sense that its generalized Fourier coefficients satisfy the bounds

$$\Big|\int f_1(t)e^{-iq(t)}dt\Big| \le C|\lambda|^{-\tau}\|f_1\|_2 \text{ uniformly for all real-valued polynomials } q.$$

Indeed, f_1 could otherwise be decomposed in L^2 into its projection onto e^{iq} plus an orthogonal vector in such a way that

$$\|f_1 - ce^{iq}\|_2 \le (1 - |\lambda|^{-2\tau})\|f_1\|_2,$$

and the desired conclusion would then follow from this and the inductive hypothesis.

We lose no generality in assuming that $\|f_j\|_2 \le 1$ for every j.

Endowing $\mathbb{R}^m$ with suitable coordinates is once again an important technical point of the proof. For this purpose, let e_1 be a unit vector orthogonal to the span of $\{V_j\}_{j=2}^m$, and e_2 be a unit vector orthogonal to the span of $\{V_j\}_{j=m+1}^n$ and not orthogonal to V_1. Then e_1 and e_2 are linearly independent, and so we write $\mathbb{R}^m \ni x = t_1e_1 + t_2e_2 + y$ (that is, $x = (t, y) \in \mathbb{R}^2 \times \mathbb{R}^{m-2}$).

Defining $P^y(t) := P(t, y)$,

$$F_1^y(t_2) := \prod_{j=2}^m f_j(\pi_j(t,y)), \quad F_2^y(t_1) := \prod_{j=m+1}^n f_j(\pi_j(t,y)) \quad \text{and} \quad G^y(\pi(t)) := f_1(\pi_1(t,y)),$$

we have that

$$\Lambda_\lambda(f_1,\dots,f_n) = \int_{\mathbb{R}^{m-2}}\Big(\int_{\mathbb{R}^2} e^{i\lambda P^y(t)}F_1^y(t_2)F_2^y(t_1)G^y(\pi(t))\eta(t,y)dt\Big)dy =: \int_{\mathbb{R}^{m-2}}\Lambda_\lambda^y dy.$$

Moreover the assumptions, Fubini's theorem and Cauchy-Schwarz together imply that

$$\int_{\mathbb{R}^{m-2}}\|F_1^y\|_2\|F_2^y\|_2\|G^y\|_2 dy < \infty. \tag{3}$$

The last important step consists of introducing a set of "bad" parameters, denoted $\mathcal{B}$, consisting of all y for which P^y has small norm in the quotient space $\mathcal{P}_{\leq d}$ modulo degenerate polynomials with respect to the three projections $t \mapsto t_1, t_2, \pi(t)$. More precisely, $y \in \mathcal{B}$ if P^y can be decomposed as

$$P^y(t) = Q_1(t_1) + Q_2(t_2) + Q_3(\pi(t)) + R(t),$$

for some polynomials Q_j and R of degree $\leq d$, with the additional requirement that $\|R\| \leq |\lambda|^{-\rho}$ (here $\|\cdot\|$ denotes a given norm on $\mathcal{P}_{\leq d}$ and ρ is a small parameter to be chosen later on).

On $\mathcal{B}^{\complement}$, we can use theorem 10 with $m = 2$ and $n = 3$, interpolate, and appeal to (3) to conclude that

$$\int_{y \notin \mathcal{B}} |\Lambda_\lambda^y| dy \leq C|\lambda|^{-(1-\rho)\tilde{\rho}} \text{ for some } \tilde{\rho} > 0.$$

Despite that fact that the set of bad parameters might have full measure (example 3), we will be in good shape if we show that if ρ is small enough, then there exists $\tilde{\epsilon} > 0$ such that

$$|\Lambda_\lambda^y| \leq C|\lambda|^{-\tilde{\epsilon}} \|F_1^y\|_2 \|F_2^y\|_2 \|G^y\|_2 \text{ uniformly for all } y \in \mathcal{B}.$$

This is a nice exercise in Fourier analysis and involves a clever use of the uniformity condition on f_1. I will omit the details for now and present them at the summer school.

References

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