ON THE SIZE OF KAKEYA SETS

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1. The Kakeya needle problem

Old Japanese knights (samurais) always had a lance to defend themselves from enemies. If attacked in a very confined space, they must know how to skillfully use and brandish their lance.

Considerations like these led the Japanese mathematician S. Kakeya to ask, in 1917,

What is the least area in the plane required to continuously rotate a needle of unit length and zero thickness around completely (i.e. by 360°*)?*

A circle of area $\frac{\pi}{4}$ will do the job. A deltoid (3 cusped hypocycloid) of area $2\pi(\frac{1}{4})^2 = \frac{\pi}{8}$ works as well. This was conjectured to be the solution of the problem for a number of years.

At the same time, in Russia, Besicovitch was working on a problem in Riemann integration:

Given a Riemann integrable function f on the plane, does there always exist a pair of orthogonal axes wrt which $\int f(x,y)dx$ exists for every y and the function $y \mapsto \int f(x,y)dx$ is Riemann integrable?

Besicovitch noticed that a counterexample would follow if he could construct a compact set $K \subset \mathbb{R}^2$ of Lebesgue measure zero containing a line segment in every direction.

To see this, let $K_0 := \{(x,y) \in K : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$ and take $f := \chi_{K_0}$. Since K contains a segment in every direction on which both K_0 and $K \setminus K_0$ are dense, there is a segment in each direction on which f is not Riemann integrable. On the other hand,

$$\mathcal{D}(f) = \mathcal{D}(\chi_{K_0}) = \partial K_0 \subset K$$

shows that $f \in \mathcal{R}(\mathbb{R}^2)$.

Only after Besicovitch left Russia in 1928 was it realized that a simple modification to his construction yielded a solution of arbitrarily small measure to the needle problem.

In what follows, a compact set (in $\mathbb{R}^n$, say) which contains a unit line segment in every direction will be called a **Kakeya set**.

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2. A MONSTER WITH MANY ARMS AND A TINY HEART

Given a rectangle $R \subset \mathbb{R}^2$ of side lengths 1 and 2^{-N}, define its **reach**, $\tilde{R}$, to be its translated version by one unit in the direction of its longest axis.

Theorem 2.1. *For every* $\epsilon > 0$ *there exist* $n = n_\epsilon$ *and rectangles* $R_1, \dots, R_{2^n}$ *of sidelength 1 and* 2^{-n} *such that*

$$\left| \cup_j R_j \right| < \epsilon$$

and $\{\tilde{R}_j\}_j$ *are mutually disjoint.*

Proof. Work here:

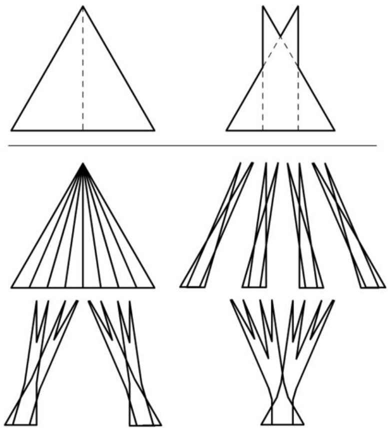

Figure 1. A 'sprouting' method for constructing a Kakeya set of small measure. Shown here are two possible ways of dividing our triangle and overlapping the pieces to get a smaller set, the first if we just use two triangles, and the second if we use eight. Notice how small the sizes are of the final figures are in comparison to each other.

Since

$$\psi_1(ABC) = \bigcup_{j=0}^{2^{n-1}-1} \tau\phi_h(A_{2j}A_{2j+2}C) \cup \bigcup_{j=0}^{2^{n-1}-1} \tau\phi_h(A_{2j}A_{2j+2}C),$$

we see that

$$|\text{heart of } \psi_1(ABC)| = \sum_{j=0}^{2^{n-1}-1} |\phi_h(A_{2j}A_{2j+2}C)| = \alpha^2|ABC|$$

and

$$|\text{arms of } \psi_1(ABC)| \leq \sum_j |\phi_a(A_{2j}A_{2j+2}C)| = 2(1-\alpha)^2|ABC|,$$

and therefore

$$|\psi_1(ABC)| \leq (\alpha^2 + 2(1-\alpha)^2)|ABC|.$$

Repeat this process in the heart of $\psi_1(ABC)$ (which consists of 2^{n-1} disjoint triangles). Retranslate all 2^n original triangles to obtain $\psi_2(ABC)$, whose heart has measure $\alpha^2\alpha^2|ABC|$ and whose new arms don't exceed $2(1-\alpha)^2\alpha^2|ABC|$ in area.

We can perform this operation n times, and at the end we will have $\psi_n(ABC)$ such that

$$\begin{aligned}|\psi_n(ABC)| &\leq (\alpha^{2n} + 2(1-\alpha)^2 + 2(1-\alpha)^2\alpha^2 + \ldots + 2(1-\alpha)^2\alpha^{2n-2})|ABC| \\ &= (\alpha^{2n} + 2(1-\alpha)^2(1+\alpha^2+\ldots+\alpha^{2n-2}))|ABC| \\ &\leq (\alpha^{2n} + 2(1-\alpha))|ABC|.\end{aligned}$$

If we take $\alpha \sim 1$ and n large, the area of $\psi_n(ABC)$ can be made arbitrarily small. Also, $\psi_n(ABC)$ contains line segments of unit length in all directions at angles at least 60° with AB.

Finally, move from triangles to rectangles.

Recall why $T^*_{j_1}$ and $T^*_{j_2}$ will never overlap at later stages of the process.

We have 2^n rectangles of sidelengths 1 and 2^{-N} for some $N = n + c$ (c large). Take 2^c disjoint translates, and we are done! □

3. Applications to harmonic analysis

3.1. **Counterexample for rectangles with arbitrary orientation.** Let $K_\epsilon := \cup_j R_j$ be the set of measure $< \epsilon$ constructed in the previous section. There are a number of problems in harmonic analysis in which K_ϵ plays a prominent role, and we now present two of them.

Problem 3.1. *Given a collection of sets $\mathcal{C} = \{C\}$ in $\mathbb{R}^n$, for what classes of functions do we have that*

$$\lim_{diam(C)\to 0} \frac{1}{|C|}\int_C f(x-y)dy = f(x) \quad x\text{-a.e.?}$$

Example 3.1. *(Lebesgue differentiation theorem) For $f \in L^1_{loc}(\mathbb{R}^n)$, we have that*

$$\lim_{r\to 0} \frac{1}{|B(r)|}\int_{B(x,r)} f(y)dy = f(x) \quad x\text{-a.e.}$$

This is immediate from the weak type (1,1) estimate involving the Hardy-Littlewood maximal function.

Results of this kind are consequences of suitable estimates related to the corresponding maximal functions. Conversely, a.e. assertions imply weak type inequalities, as the next result illustrates.

Proposition 3.1. *Let $\{d\mu_j\}_{j\in\mathbb{N}}$ be a countable collection of finite nonnegative measures on $\mathbb{R}^n$, supported on a fixed compact set K. Let $p \in [1,\infty)$ and consider the maximal function*

$$\mathcal{M}f(x) := \sup_j |f * d\mu_j(x)|.$$

If, for every $f \in L^p(\mathbb{R}^n)$, $\mathcal{M}f(x) < \infty$ for some set of x having positive measure, then $f \mapsto \mathcal{M}f$ is of weak type (p,p), i.e.,

$$|\{x : \mathcal{M}f(x) > \alpha\}| \leq C\alpha^{-p}\|f\|_p^p, \quad \forall f \in L^p, \forall \alpha > 0.$$

Proof. Cf. [6], chapter X (pp. 441-4). □

Together with the construction from section 2, this proposition can be used to provide an example where Problem 3.1. has a negative answer. For this purpose, let $\mathcal{R}$ be the collection of *all* rectangles in $\mathbb{R}^2$ centered at the origin. Note that the elements of this set do not have bounded eccentricity. Then we have the following

Corollary 3.1. *For each $p \in [1, \infty)$, there exists $f \in L^p(\mathbb{R}^2)$ such that*

$$\limsup_{diam(R)\to 0} \frac{1}{|R|}\int_R f(x-y)dy = \infty, \quad x\text{-}a.e.$$

where the limit is defined as $\inf_{\delta>0}(\sup_{diam(R)<\delta, R\in\mathcal{R}})$.

Proof. Consider first

$$\mathcal{M}f(x) = \sup_{\mathrm{diam}(R)<8} \frac{1}{|R|}\Big|\int_R f(x-y)dy\Big|.$$

By restricting our attention to rectangles with rational side lengths and orientations, we see that $\mathcal{M}$ is of the form of the maximal functions considered in proposition 3.1.. By that proposition, it will be enough to show that $\mathcal{M}$ is not of weak type (p,p).

For that purpose, let $f := \chi_{K_\epsilon}$ (recall $K_\epsilon = \cup_{j=1}^{2^n} R_j$, $|K_\epsilon| < \epsilon$ and $\{\tilde{R}_j\}_j$ are disjoint). Then $\|f\|_p^p = |K_\epsilon| < \epsilon$. It will be enough to show the existence of a positive constant δ such that $\mathcal{M}f \geq \delta > 0$ on the set $\cup_j \tilde{R}_j$ (of measure 1).

For $x \in \tilde{R}_j$, construct R (centered at x) of the same width and orientation as R_j and s.t. $|R \cap R_j| \geq \frac{|R|}{12}$.

Then

$$\mathcal{M}f(x) \geq \frac{1}{|R|}\Big|\int_{R-x} \chi_{K_\epsilon}(x-y)dy\Big| \geq \frac{|(R-x)\cap -(K_\epsilon - x)|}{|R|} \geq \frac{|R\cap R_j|}{|R|} \geq \frac{1}{12}.$$

For the general case, dilate! □

3.2. **Counterexample for the ball multiplier.** For $f \in L^2 \cap L^p(\mathbb{R}^n)$, define

$$S_R f(x) = \int_{|\xi|\leq R} \hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

Problem 3.2. *Can S_R be extended to a bounded operator on L^p such that $\|S_R f - f\|_p \to 0$ as $R \to \infty$, for all $f \in L^p$?*

The answer has long been known to be positive for $p = 2$ and negative for $p \in \{1, \infty\}$. In the one dimensional case, M. Riesz's theorem asserts that convergence holds for $p \in (1, \infty)$.

Is something similar to be expected for higher dimensions? Definitely not! This so-called "ball multiplier conjecture" was disproved by C. Fefferman in 1971. He used Kakeya sets to prove that Fourier series in higher dimensions, when summed spherically, do not necessarily converge in the L^p norm if $p \neq 2$.

Very roughly, his construction involves taking f to be a sum of characteristic functions of long, thin tubes multiplied by some appropriate phase factors. The operator S acts as a "shift", so that Sf can be made to resemble the characteristic funciton of a Kakeya set. The small support of Sf and Hölder inequality together imply that $\|Sf\|_p$ is large.

4. The Kakeya conjecture

We have seen that Kakeya sets can have arbitrarily small Lebesgue measure. Actually, further modifications to the construction presented in section 1 allow the construction of a Kakeya set of zero Lebesgue measure. This still doesn't settle everything about the "size" of Kakeya sets.

Recall the definition of Hausdorff measure of a Borel subset E of a metric space X: if

$$m_\beta(E) = \lim_{\delta\to 0} \inf \Big\{ \sum_k (\mathrm{diam} F_k)^\beta : E \subset \bigcup_{k=1}^{\infty} F_k, \mathrm{diam} F_k \leq \delta, \forall k \Big\},$$

then $\alpha = \sup\{\beta : m_\beta(E) = \infty\} = \inf\{\beta : m_\beta(E) = 0\}$ is the Hausdorff dimension of E. We write $Haus(E) = \alpha$.

Example 4.1. *in what follows, $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{P}$ denote, respectively, the Cantor set, the Sierpiński triangle and the Peano curve in $\mathbb{R}^2$.*

i) $Haus(\mathbb{R}^n) = n$;
ii) $Haus(\mathbb{T}) = 1$;
iii) $Haus(\{x_1, x_2, \ldots\}) = 0$;
iv) $Haus(\mathcal{C}) = \frac{\log 2}{\log 3}$;
v) $Haus(\mathcal{S}) = \frac{\log 3}{\log 2}$;
vi) $Haus(\mathcal{P}) = 2$.

What can we say about the Hausdorff dimension of Kakeya sets?

Conjecture 4.1. *The Hausdorff dimension of a Kakeya set in $\mathbb{R}^n$ is n.*

4.1. What is known.

In 1971, Davies proved that if $K \subset \mathbb{R}^2$ is a Kakeya set, then $Haus(K) = 2$. For $n \geq 3$, the result is still open. Some partial results are known:

(Wolff, 94) If $n = 3$, then $Haus(K) \geq \frac{5}{2}$;
(Wolff, 94) If $n = 4$, then $Haus(K) \geq 3$;
(Katz-Tao, 2001) For $n \geq 5$, $Haus(K) \geq (2-\sqrt{2})(n-4)+3$.

4.2. The finite field Kakeya conjecture.

In 1999, T. Wolff proposed a simpler finite field analogue for the Kakeya conjecture as a model problem that avoided all technical issues related to the evasive concept of Hausdorff dimension.

Let $\mathbb{F}$ be a finite field. We say that $K \subset \mathbb{F}^n$ is a **Kakeya set** if

$$\forall x \in \mathbb{F}^n, \exists y \in \mathbb{F}^n \text{ s.t. the line } L_{x,y} := \{y + a \cdot x | a \in \mathbb{F}\} \text{ is contained in } K.$$

Conjecture 4.2. *Let* $K \subset \mathbb{F}^n$ *be a Kakeya set. Then* $|K| \geq c_n|\mathbb{F}|^n$ *(*c_n *depends only on* n*).*

This is no longer a conjecture. It was proved by Z. Dvir in a paper uploaded to arXiv in March 2008 (cf.[2]), via a "beautifully simple argument" (cf. Tao's blog) that uses the polynomial method from algebraic extremal combinatorics. This provides higher dimensional analogues of the following basic facts:

i) If $0 \neq p \in \mathbb{F}[x]$ has degree at most d, then $|\{x \in \mathbb{F} : p(x) = 0\}| \leq d$ (use division algorithm).

ii) If $E \subset \mathbb{F}$ is such that $|E| \leq d$, then there exists a nonzero polynomial $p \in \mathbb{F}[x]$ of degree at most d such that $p(x) = 0, \forall x \in E$ (if $E = \{x_1, \ldots, x_d\}$, take $p(x) = (x - x_1) \ldots (x - x_d)$).

The following is a 2-dimensional analogue of ii):

Lemma 4.1. *Let* $E \subset \mathbb{F}^n$ *be such that* $|E| < \binom{n+d}{n}$ $\boxed{\text{for some } 0 \leq d < |\mathbb{F}|}$. *Then there exists a nonzero polynomial* $p \in \mathbb{F}[x_1, \ldots, x_n]$ *of degree at most* d *such that* $p(x) = 0$*, for every* $x \in E$*.*

Proof. Let V be the vector space of all polynomials $p \in \mathbb{F}[x_1, \ldots, x_n]$ of degree $\leq d$. Since $d < |\mathbb{F}|$, $\dim V = \binom{n+d}{n}$ (apples and dividers). On the other hand, $\dim F^E = |E| < \binom{n+d}{n}$, so the (linear) map

$$\begin{array}{rcl} V & \to & F^E \\ p & \mapsto & (p(x))_{x \in E} \end{array}$$

cannot be injective. □

Lemma 4.2. *Let* $K \subset \mathbb{F}^n$ *be a Kakeya set and* $p \in \mathbb{F}[x_1, \ldots, x_n]$ *a polynomial of degree less than* $|\mathbb{F}|$ *such that* $p(x) = 0$*, for every* $x \in K$*. Then* $p = 0$ *(as polynomials).*

Proof. We proceed by contradiction. Suppose $p \neq 0$. Then, writing $p = \sum_{i=0}^{d} p_i$ as the sum of its homogeneous components ($0 \leq d := \deg p \leq |\mathbb{F}| - 1$), we see that $d > 0$ (since p vanishes on K).

Pick any $v \in \mathbb{F}^n \setminus \{0\}$. Let $x = x_v$ be such that $L_{v,x} \subset K$. Then

$$p(x + t \cdot v) = 0, \quad \forall t \in \mathbb{F}.$$

The LHS of the last equation is a polynomial in t of degree $d \leq |\mathbb{F}| - 1$. In particular (factor theorem), $p_d(v) = t^d$ coefficient of $p(x + t \cdot v) = 0, \forall v \in \mathbb{F}^n \setminus \{0\}$. Since p_d is homogeneous and $d > 0$, p_d vanishes on all of $\mathbb{F}^n$. Use i) n times (once for each variable) to conclude $p_d = 0$, which is the desired contradiction. □

The last step in the above argument can be alternatively justified as follows: if $p \in \mathbb{F}[x_1, \ldots, x_n]$ is such that $\deg_{x_i} p < |\mathbb{F}|$ for each $i \leq n$, and p induces the zero function on $\mathbb{F}^n$, then $p = 0$ (cf. Lang, p. 177).

An immediate consequence of two preceding lemmas is the following

Corollary 4.1. *Every Kakeya set in* $\mathbb{F}^n$ *has cardinality at least* $\binom{|\mathbb{F}|+n-1}{n}$.

Since

$$\binom{|\mathbb{F}|+n-1}{n} = \frac{1}{n!}|\mathbb{F}|^n + O_n(|\mathbb{F}|^{n-1}),$$

the Kakeya conjecture for finite fields is established.

A few final remarks:

- i) Szeméredi-Trotter is false for finite fields (cf. [8], p.139).
- ii) The polynomial method is extremely dependent of the algebraic nature of the finite field setting and does not seem to extend directly to the Euclidean case. However, the result lends significant indirect support to the Euclidean case.
- iii) The constants $c_n(= \frac{1}{n!})$ in the Kakeya conjecture for finite fields are not uniform in n. This is unfortunate, but otherwise the generalization for the Euclidean case would be straightforward. On the other hand, in a paper from August 2008, S. Saraf and M. Sudan were able to pin down the growth of the leading constant by showing that the size of a Kakeya set in $\mathbb{F}^n$ is at least $c_n|\mathbb{F}|^n$ where $c_n \sim \beta^n$ for a constant $\beta > 0$. Cf. [5].

References

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