L^p estimates for the Hilbert transform along a one-variable vector field

Diogo Oliveira e Silva
(after M. Bateman and C. Thiele)

Summer School in Geometric and Fourier analytic questions in Euclidean space, Lake Arrowhead, CA
Sep 12, 2011

The Hilbert transform

$$Hf(x) := \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(x-t)}{t} dt.$$

The Hilbert transform

$$\boxed{Hf(x) := \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(x-t)}{t} dt.}$$

Theorem

For $f \in \mathcal{S}(\mathbb{R})$ and $1 < p < \infty$,

The Hilbert transform

$$Hf(x) := \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(x-t)}{t} dt.$$

Theorem

For $f \in \mathcal{S}(\mathbb{R})$ and $1 < p < \infty$,

- *Riesz (1924):*

$$\|Hf\|_p \leq C_p \|f\|_p;$$

The Hilbert transform

$$\boxed{Hf(x) := \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(x-t)}{t}\, dt.}$$

Theorem

For $f \in \mathcal{S}(\mathbb{R})$ and $1 < p < \infty$,

- *Riesz (1924):*

$$\|Hf\|_p \leq C_p \|f\|_p;$$

- *Kolmogorov (1925):*

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq \frac{C}{\lambda} \|f\|_1.$$

The Hilbert transform on L^1/L^∞ and L^2

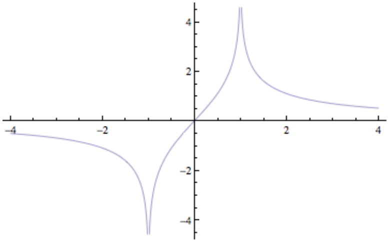
- The strong (p,p) inequality is false for $p=1$ or $p=\infty$: if $f = 1_{[-1,1]}$, then

$$Hf(x) = \frac{1}{\pi}\log\left|\frac{x+1}{x-1}\right|.$$

The Hilbert transform on L^1/L^∞ and L^2

- The strong (p,p) inequality is false for $p=1$ or $p=\infty$: if $f = 1_{[-1,1]}$, then

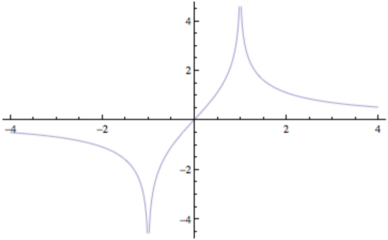
$$Hf(x) = \frac{1}{\pi}\log\left|\frac{x+1}{x-1}\right|.$$



The Hilbert transform on L^1/L^∞ and L^2

- The strong (p, p) inequality is false for $p = 1$ or $p = \infty$: if $f = 1_{[-1,1]}$, then

$$Hf(x) = \frac{1}{\pi} \log \left| \frac{x+1}{x-1} \right|.$$



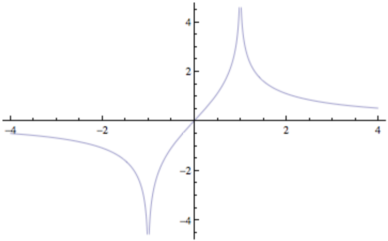
- On the Fourier side,

$$\widehat{Hf}(\xi) = -i \operatorname{sgn}(\xi) \widehat{f}(\xi).$$

The Hilbert transform on L^1/L^∞ and L^2

- The strong (p,p) inequality is false for $p=1$ or $p=\infty$: if $f = 1_{[-1,1]}$, then

$$Hf(x) = \frac{1}{\pi}\log\left|\frac{x+1}{x-1}\right|.$$



- On the Fourier side,

$$\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi).$$

It follows that $\|Hf\|_2 = \|f\|_2$.

The Hilbert transform along a vector field

$$\boxed{H_v f(x,y) := p.v. \int_{\mathbb{R}} \frac{f((x,y) - tv(x,y))}{t} dt.}$$

The Hilbert transform along a vector field

$$\boxed{H_v f(x,y) := p.v. \int_{\mathbb{R}} \frac{f((x,y) - tv(x,y))}{t} dt.}$$

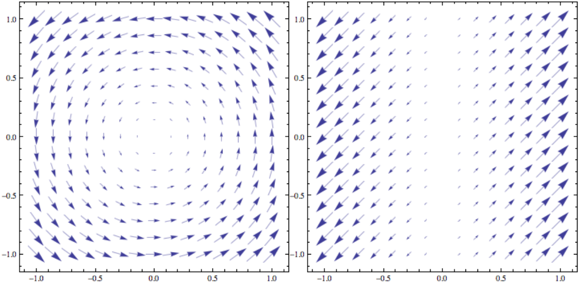
This is a one-dimensional transform along a particular line in the plane, which varies from point to point.

The Hilbert transform along a vector field

$$H_v f(x,y) := p.v. \int_{\mathbb{R}} \frac{f((x,y) - tv(x,y))}{t} dt.$$

This is a one-dimensional transform along a particular line in the plane, which varies from point to point.

Figure: $v_1(x,y) = (-y, x)$ and $v_2(x,y) = (x,x) = v_2(x,0)$



Besicovitch (counter)example

Figure: Intermediate step in the construction of a "tiny" compact set of the plane which contains a line segment of unit length in each direction.

Besicovitch (counter)example

Figure: Intermediate step in the construction of a "tiny" compact set of the plane which contains a line segment of unit length in each direction.

Given $\epsilon > 0$ one can select "rectangles" $R_1, \ldots, R_n$ with disjoint reaches and of unit length such that $|B| \leq \epsilon$ for $B := \bigcup_{j=1}^{n} R_j$.

Besicovitch (counter)example

Figure: Intermediate step in the construction of a "tiny" compact set of the plane which contains a line segment of unit length in each direction.

Given $\epsilon > 0$ one can select "rectangles" $R_1, \ldots, R_n$ with disjoint reaches and of unit length such that $|B| \leq \epsilon$ for $B := \bigcup_{j=1}^{n} R_j$. On $\mathrm{Reach(B)}$ define a vector field v which points to a line segment contained in B.

Besicovitch (counter)example

Figure: Intermediate step in the construction of a "tiny" compact set of the plane which contains a line segment of unit length in each direction.

Given $\epsilon > 0$ one can select "rectangles" $R_1, \ldots, R_n$ with disjoint reaches and of unit length such that $|B| \leq \epsilon$ for $B := \bigcup_{j=1}^{n} R_j$. On $\mathrm{Reach(B)}$ define a vector field v which points to a line segment contained in B. Then

$$|H_v 1_B(x)| \approx 1 \text{ if } x \in \mathrm{Reach(B)}.$$

Main result

Theorem [M. Bateman and C. Thiele (2011)]

Let:

- $\frac{3}{2} < p < \infty$;

Main result

Theorem [M. Bateman and C. Thiele (2011)]

Let:

- $\frac{3}{2} < p < \infty$;
- v *nonvanishing, measurable;*

Main result

Theorem [M. Bateman and C. Thiele (2011)]

Let:

- $\frac{3}{2} < p < \infty$;
- v *nonvanishing, measurable;*
- $v(x, y) = v(x, 0)$.

Main result

Theorem [M. Bateman and C. Thiele (2011)]

Let:

- $\frac{3}{2} < p < \infty$;
- v *nonvanishing, measurable;*
- $v(x, y) = v(x, 0)$.

Then

$$\|H_v f\|_p \lesssim \|f\|_p \text{ for every } f \in L^p(\mathbb{R}^2).$$

A few remarks

- The result for constant vector fields follows from the L^p boundedness of H.

A few remarks

- The result for constant vector fields follows from the L^p boundedness of H.
- May assume unit length vector field.

A few remarks

- The result for constant vector fields follows from the L^p boundedness of H.
- May assume unit length vector field.
- Enough to consider vector fields with nonvanishing first component.

A few remarks

- The result for constant vector fields follows from the L^p boundedness of H.
- May assume unit length vector field.
- Enough to consider vector fields with nonvanishing first component.
- Actually,

$$v(x,y) = (1, u(x)) \text{ with } \|u\|_\infty \leq 10^{-2}$$

will do (due to symmetries).

A few remarks

- The result for constant vector fields follows from the L^p boundedness of H.
- May assume unit length vector field.
- Enough to consider vector fields with nonvanishing first component.
- Actually,

$$v(x, y) = (1, u(x)) \text{ with } \|u\|_\infty \leq 10^{-2}$$

  will do (due to symmetries).
- Sharpness of the endpoint exponent $p = \frac{3}{2}$ is open.

“On convergence and growth of partial sums of Fourier series”

"On convergence and growth of partial sums of Fourier series"

Theorem [L. Carleson (1966)]

For any measurable function $\eta : \mathbb{R} \to \mathbb{R}$, *the operator*

$$\boxed{C_\eta f(x) := p.v. \int_{\mathbb{R}} f(x-t) e^{i\eta(x)t} \frac{dt}{t}}$$

maps $L^2(\mathbb{R})$ *into itself, with operator norm independent of* η.

“On convergence and growth of partial sums of Fourier series”

Theorem [L. Carleson (1966)]

For any measurable function $\eta : \mathbb{R} \to \mathbb{R}$, *the operator*

$$\boxed{C_\eta f(x) := p.v. \int_{\mathbb{R}} f(x-t) e^{i\eta(x)t} \frac{dt}{t}}$$

maps $L^2(\mathbb{R})$ *into itself, with operator norm independent of* η.

Corollary

For every $f \in L^2(\mathbb{R})$ *we have*

$$\lim_{N\to\infty} \int_{|\xi|\le N} \widehat{f}(\xi) e^{ix\xi} d\xi = f(x) \ \ x\text{-}a.e.$$

"On convergence and growth of partial sums of Fourier series"

Theorem [L. Carleson (1966)]

For any measurable function $\eta : \mathbb{R} \to \mathbb{R}$*, the operator*

$$\boxed{C_\eta f(x) := p.v. \int_{\mathbb{R}} f(x-t) e^{i\eta(x)t} \frac{dt}{t}}$$

maps $L^2(\mathbb{R})$ *into itself, with operator norm independent of* η*.*

Corollary

For every $f \in L^2(\mathbb{R})$ *we have*

$$\lim_{N\to\infty} \int_{|\xi|\leq N} \widehat{f}(\xi) e^{ix\xi} d\xi = f(x) \;\; x\text{-}a.e.$$

- The case $p = 2$ of our main theorem is equivalent to Carleson's theorem.

Coifman's argument: Carleson $\Rightarrow p = 2$ ✔

$$\left\| \int_{\mathbb{R}} \frac{f(x-t, y-tu(x))}{t} dt \right\|_{L^2_{x,y}}$$

Coifman's argument: Carleson $\Rightarrow p = 2$ ✔

$$\Big\| \int_{\mathbb{R}} \frac{f(x-t, y-tu(x))}{t} dt \Big\|_{L^2_{x,y}} = \Big\| \int e^{iy\xi} \int \widehat{f}(x-t,\xi) e^{-itu(x)\xi} \frac{dt}{t} d\xi \Big\|_{L^2_{x,y}}$$

Coifman's argument: Carleson $\Rightarrow p = 2$ ✔

$$\left\| \int_{\mathbb{R}} \frac{f(x-t, y-tu(x))}{t} dt \right\|_{L^2_{x,y}} = \left\| \int e^{iy\xi} \int \widehat{f}(x-t,\xi) e^{-itu(x)\xi} \frac{dt}{t} d\xi \right\|_{L^2_{x,y}}$$

$$= \left\| \int \widehat{f}(x-t,\xi) e^{-itu(x)\xi} \frac{dt}{t} \right\|_{L^2_{x,\xi}}$$

Coifman's argument: Carleson $\Rightarrow p = 2$ ✔

$$\Big\| \int_{\mathbb{R}} \frac{f(x-t, y-tu(x))}{t} dt \Big\|_{L^2_{x,y}} = \Big\| \int e^{iy\xi} \int \widehat{f}(x-t,\xi) e^{-itu(x)\xi} \frac{dt}{t} d\xi \Big\|_{L^2_{x,y}}$$

$$= \Big\| \int \widehat{f}(x-t,\xi) e^{-itu(x)\xi} \frac{dt}{t} \Big\|_{L^2_{x,\xi}}$$

$$\lesssim \|\widehat{f}\|_2$$

Coifman's argument: Carleson $\Rightarrow p = 2$ ✔

$$
\begin{aligned}
\Big\| \int_{\mathbb{R}} \frac{f(x-t, y-tu(x))}{t}\, dt \Big\|_{L^2_{x,y}} &= \Big\| \int e^{iy\xi} \int \widehat{f}(x-t,\xi) e^{-itu(x)\xi} \frac{dt}{t}\, d\xi \Big\|_{L^2_{x,y}} \\
&= \Big\| \int \widehat{f}(x-t,\xi) e^{-itu(x)\xi} \frac{dt}{t} \Big\|_{L^2_{x,\xi}} \\
&\lesssim \|\widehat{f}\|_2 = \|f\|_2 .
\end{aligned}
$$

Coifman's argument: Carleson $\Rightarrow p = 2$ ✔

$$\left\| \int_{\mathbb{R}} \frac{f(x-t, y-tu(x))}{t} dt \right\|_{L^2_{x,y}} = \left\| \int e^{iy\xi} \int \widehat{f}(x-t,\xi) e^{-itu(x)\xi} \frac{dt}{t} d\xi \right\|_{L^2_{x,y}}$$

$$= \left\| \int \widehat{f}(x-t,\xi) e^{-itu(x)\xi} \frac{dt}{t} \right\|_{L^2_{x,\xi}}$$

$$\lesssim \|\widehat{f}\|_2 = \|f\|_2.$$

We used Fourier inversion, Plancherel and Carleson.

Simplified general case: truncate (at $|t| \leq \epsilon$), normalize ($\|v\|_\infty \lesssim 1$) and demand slow rotation ($\|Dv\|_\infty \lesssim 1$)

$$\boxed{H_{v,\epsilon} f(\vec{x}) := p.v. \int_{-\epsilon}^{\epsilon} \frac{f(\vec{x} - tv(\vec{x}))}{t} dt.}$$

Simplified general case: truncate (at $|t| \leq \epsilon$), normalize ($\|v\|_\infty \lesssim 1$) and demand slow rotation ($\|Dv\|_\infty \lesssim 1$)

$$H_{v,\epsilon} f(\vec{x}) := p.v. \int_{-\epsilon}^{\epsilon} \frac{f(\vec{x} - tv(\vec{x}))}{t} dt.$$

Conjecture 1 [Stein]

If $\|v\|_{Lip}$ is sufficiently small (dep. on ϵ), then $H_{v,\epsilon}$ is bounded on L^2.

Simplified general case: truncate (at $|t| \leq \epsilon$), normalize ($\|v\|_\infty \lesssim 1$) and demand slow rotation ($\|Dv\|_\infty \lesssim 1$)

$$H_{v,\epsilon} f(\vec{x}) := p.v. \int_{-\epsilon}^{\epsilon} \frac{f(\vec{x} - tv(\vec{x}))}{t} dt.$$

Conjecture 1 [Stein]

If $\|v\|_{Lip}$ is sufficiently small (dep. on ϵ), then $H_{v,\epsilon}$ is bounded on L^2.

$$M_{v,\epsilon} f(x) := \sup_{0<s\leq\epsilon} (2s)^{-1} \int_{-s}^{s} |f(\vec{x} - tv(\vec{x}))| dt.$$

Simplified general case: truncate (at $|t| \leq \epsilon$), normalize ($\|v\|_\infty \lesssim 1$) and demand slow rotation ($\|Dv\|_\infty \lesssim 1$)

$$\boxed{H_{v,\epsilon} f(\vec{x}) := p.v. \int_{-\epsilon}^{\epsilon} \frac{f(\vec{x} - tv(\vec{x}))}{t} dt.}$$

Conjecture 1 [Stein]

If $\|v\|_{Lip}$ is sufficiently small (dep. on ϵ), then $H_{v,\epsilon}$ is bounded on L^2.

$$\boxed{M_{v,\epsilon} f(x) := \sup_{0<s\leq\epsilon} (2s)^{-1} \int_{-s}^{s} |f(\vec{x} - tv(\vec{x}))| dt.}$$

Conjecture 2 [Zygmund]

If $\|v\|_{Lip}$ is sufficiently small (dep. on ϵ), then $M_{v,\epsilon}$ is bounded on L^2.

Related results

If v is real-analytic and the operator is restricted to a bounded domain, then:

Related results

If v is real-analytic and the operator is restricted to a bounded domain, then:

- Bourgain (1989) [Stefan]: MO is bounded on L^p for $p > 1$;

Related results

If v is real-analytic and the operator is restricted to a bounded domain, then:

- Bourgain (1989) [Stefan]: MO is bounded on L^p for $p > 1$;
- Christ, Nagel, Stein, Wainger (1999): Same result for HT, assuming no integral curve of the vector field forms a straight line;

Related results

If v is real-analytic and the operator is restricted to a bounded domain, then:

- Bourgain (1989) [Stefan]: MO is bounded on L^p for $p > 1$;
- Christ, Nagel, Stein, Wainger (1999): Same result for HT, assuming no integral curve of the vector field forms a straight line;
- Stein, Street (Preprint, 2010): HT is bounded on L^p for $p > 1$.

A companion result (M. Bateman, *Single Annulus L^p estimates for Hilbert transforms along vector fields*. Preprint, 2011.)

Theorem

Assume $\|u\|_\infty \le 1$ *and* $p \in (1, \infty)$ *and* $\widehat{f}(\xi, \eta)$ *vanishes outside an annulus* $A < |(\xi, \eta)| < 2A$ *for some* $A > 0$*. Then*

$$\|H_v f\|_p \lesssim \|f\|_p.$$

A companion result (M. Bateman, *Single Annulus L^p estimates for Hilbert transforms along vector fields*. Preprint, 2011.)

Theorem

Assume $\|u\|_\infty \leq 1$ *and* $p \in (1,\infty)$ *and* $\widehat{f}(\xi,\eta)$ *vanishes outside an annulus* $A < |(\xi,\eta)| < 2A$ *for some* $A > 0$*. Then*

$$\|H_v f\|_p \lesssim \|f\|_p.$$

Equivalently (using the symmetries discussed in the next slide):

A companion result (M. Bateman, *Single Annulus L^p estimates for Hilbert transforms along vector fields*. Preprint, 2011.)

Theorem

Assume $\|u\|_\infty \leq 1$ and $p \in (1, \infty)$ and $\widehat{f}(\xi, \eta)$ vanishes outside an annulus $A < |(\xi, \eta)| < 2A$ for some $A > 0$. Then

$$\|H_v f\|_p \lesssim \|f\|_p.$$

Equivalently (using the symmetries discussed in the next slide):

Theorem

Assume $p \in (1, \infty)$ and $\widehat{f}(\xi, \eta)$ is supported on a horizontal pair of strips $A < |\eta| < 2A$ for some $A > 0$. Then

$$\|H_v f\|_p \lesssim \|f\|_p.$$

Symmetries

Vector fields depending on the first variable are invariant under linear transformations which preserve the vertical direction:

Symmetries

Vector fields depending on the first variable are invariant under linear transformations which preserve the vertical direction:

- Isotropic dilations: $(x, y) \mapsto (\lambda x, \lambda y)$

Symmetries

Vector fields depending on the first variable are invariant under linear transformations which preserve the vertical direction:

- Isotropic dilations: $(x, y) \mapsto (\lambda x, \lambda y)$

- Dilations of the second variable: $(x, y) \mapsto (x, \lambda y)$

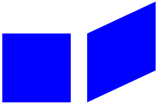
Symmetries

Vector fields depending on the first variable are invariant under linear transformations which preserve the vertical direction:

- Isotropic dilations: $(x, y) \mapsto (\lambda x, \lambda y)$

- Dilations of the second variable: $(x, y) \mapsto (x, \lambda y)$

- Shearing transformations: $(x, y) \mapsto (x, y + \lambda x)$



The main approach

Instead of looking directly at H_v, consider

$$\boxed{H_k := P_k H_v P_c}$$

The main approach

Instead of looking directly at H_v, consider

$$\boxed{H_k := P_k H_v P_c}$$

where

$$\boxed{\widehat{P_c f}(\xi,\eta) := 1_{10|\xi|\le|\eta|}\widehat{f}(\xi,\eta)}$$

The main approach

Instead of looking directly at H_v, consider

$$\boxed{H_k := P_k H_v P_c}$$

where

$$\boxed{\widehat{P_c f}(\xi,\eta) := 1_{10|\xi|\leq|\eta|}\widehat{f}(\xi,\eta)}$$

and

$$\boxed{\widehat{P_k f} := 1_{B_k}\widehat{f}} \text{ with } B_k := \{(\xi,\eta) : 2^k \leq |\eta| < 2^{k+.01}\}.$$

The main approach

Instead of looking directly at H_v, consider

$$\boxed{H_k := P_k H_v P_c}$$

where

$$\boxed{\widehat{P_c f}(\xi,\eta) := 1_{10|\xi|\le|\eta|}\widehat{f}(\xi,\eta)}$$

and

$$\boxed{\widehat{P_k f} := 1_{B_k}\widehat{f}} \text{ with } B_k := \{(\xi,\eta) : 2^k \le |\eta| < 2^{k+.01}\}.$$

- ETE $H_v P_c$ instead of H_v because $\|u\|_\infty \le .01$ implies

$$H_v(1-P_c) = H_{(1,0)}(1-P_c).$$

The main approach

Instead of looking directly at H_v, consider

$$\boxed{H_k := P_k H_v P_c}$$

where

$$\boxed{\widehat{P_c f}(\xi,\eta) := 1_{10|\xi|\leq|\eta|}\widehat{f}(\xi,\eta)}$$

and

$$\boxed{\widehat{P_k f} := 1_{B_k}\widehat{f}} \text{ with } B_k := \{(\xi,\eta) : 2^k \leq |\eta| < 2^{k+.01}\}.$$

- ETE $H_v P_c$ instead of H_v because $\|u\|_\infty \leq .01$ implies

$$H_v(1-P_c) = H_{(1,0)}(1-P_c).$$

- The projections P_k will allow us to use the Companion Theorem.

The one-variable property of v

Since

The one-variable property of v

Since

- HT in constant direction is given by a Fourier multiplier, and

The one-variable property of v

Since

- HT in constant direction is given by a Fourier multiplier, and
- v is constant along vertical lines,

The one-variable property of v

Since

- HT in constant direction is given by a Fourier multiplier, and
- v is constant along vertical lines,

we have that (formally)

$$H_v f(x,y) = \iint m_x(\xi,\eta)\widehat{f}(\xi,\eta)e^{i(x,y)\cdot(\xi,\eta)}d\xi d\eta.$$

The one-variable property of v

Since

- HT in constant direction is given by a Fourier multiplier, and
- v is constant along vertical lines,

we have that (formally)

$$H_v f(x,y) = \iint m_x(\xi,\eta)\widehat{f}(\xi,\eta)e^{i(x,y)\cdot(\xi,\eta)}d\xi d\eta.$$

It follows that

$$H_v(P_k f)(x,y) = \int 1_{[2^k,2^{k+.01})}(|\eta|)e^{iy\eta}\int m_x(\xi,\eta)\widehat{f}(\xi,\eta)e^{ix\xi}d\xi d\eta$$

The one-variable property of v

Since

- HT in constant direction is given by a Fourier multiplier, and
- v is constant along vertical lines,

we have that (formally)

$$H_v f(x,y) = \iint m_x(\xi,\eta)\widehat{f}(\xi,\eta)e^{i(x,y)\cdot(\xi,\eta)}d\xi d\eta.$$

It follows that

$$\begin{aligned} H_v(P_k f)(x,y) &= \int 1_{[2^k,2^{k+.01})}(|\eta|)e^{iy\eta}\int m_x(\xi,\eta)\widehat{f}(\xi,\eta)e^{ix\xi}d\xi d\eta \\ &= P_k(H_v f)(x,y). \end{aligned}$$

The first estimate

$$\|H_v P_c f\|_p$$

The first estimate

$$\|H_v P_c f\|_p \lesssim \Big\| \Big(\sum_{k \in \mathbb{Z}/100} |H_k f|^2 \Big)^{1/2} \Big\|_p$$

The first estimate

$$\begin{aligned}\|H_v P_c f\|_p &\lesssim \Big\| \Big(\sum_{k \in \mathbb{Z}/100} |H_k f|^2 \Big)^{1/2} \Big\|_p \\ &= \Big\| \Big(\sum_{k \in \mathbb{Z}/100} |H_k (P_k f)|^2 \Big)^{1/2} \Big\|_p\end{aligned}$$

The first estimate

$$\begin{aligned}
\|H_v P_c f\|_p &\lesssim \Big\| \Big(\sum_{k \in \mathbb{Z}/100} |H_k f|^2 \Big)^{1/2} \Big\|_p \\
&= \Big\| \Big(\sum_{k \in \mathbb{Z}/100} |H_k (P_k f)|^2 \Big)^{1/2} \Big\|_p \\
&\lesssim \Big\| \Big(\sum_{k \in \mathbb{Z}/100} |P_k f|^2 \Big)^{1/2} \Big\|_p
\end{aligned}$$

The first estimate

$$\begin{aligned}\|H_v P_c f\|_p &\lesssim \Big\|\Big(\sum_{k\in\mathbb{Z}/100} |H_k f|^2\Big)^{1/2}\Big\|_p \\ &= \Big\|\Big(\sum_{k\in\mathbb{Z}/100} |H_k(P_k f)|^2\Big)^{1/2}\Big\|_p \\ &\lesssim \Big\|\Big(\sum_{k\in\mathbb{Z}/100} |P_k f|^2\Big)^{1/2}\Big\|_p \lesssim \|f\|_p.\end{aligned}$$

The first estimate

$$
\begin{aligned}
\|H_v P_c f\|_p &\lesssim \Big\| \Big(\sum_{k \in \mathbb{Z}/100} |H_k f|^2 \Big)^{1/2} \Big\|_p \\
&= \Big\| \Big(\sum_{k \in \mathbb{Z}/100} |H_k (P_k f)|^2 \Big)^{1/2} \Big\|_p \\
&\lesssim \Big\| \Big(\sum_{k \in \mathbb{Z}/100} |P_k f|^2 \Big)^{1/2} \Big\|_p \lesssim \|f\|_p .
\end{aligned}
$$

This is a consequence of LP theory and the fact that $[H_v, P_k] = 0$.
But something still needs to be done...

After a limiting argument...

It will be enough to prove that, for all $k_0 > 0$,

$$\Big\| \Big(\sum_{|k| \le k_0} |H_k f_k|^2 \Big)^{\frac{1}{2}} \Big\|_p \lesssim \Big\| \Big(\sum_{|k| \le k_0} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_p$$

holds for any sequence of functions $f_k \in L^2$, with implicit constant independent of k_0.

After a limiting argument...

It will be enough to prove that, for all $k_0 > 0$,

$$\Big\| \Big(\sum_{|k| \leq k_0} |H_k f_k|^2 \Big)^{\frac{1}{2}} \Big\|_p \lesssim \Big\| \Big(\sum_{|k| \leq k_0} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_p$$

holds for any sequence of functions $f_k \in L^2$, with implicit constant independent of k_0.

- Companion Theorem $\Rightarrow p = 2$ ✔;

After a limiting argument...

It will be enough to prove that, for all $k_0 > 0$,

$$\Big\| \Big(\sum_{|k| \leq k_0} |H_k f_k|^2 \Big)^{\frac{1}{2}} \Big\|_p \lesssim \Big\| \Big(\sum_{|k| \leq k_0} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_p$$

holds for any sequence of functions $f_k \in L^2$, with implicit constant independent of k_0.

- Companion Theorem $\Rightarrow p = 2$ ✔;
- Other values of p: use restricted weak-type estimates.

Restricted type estimates

Definition

A multilinear form Λ is of restricted type $\alpha = (\alpha_1, \ldots, \alpha_n)$ ($\alpha_j \in [0, 1]$) if there exists $C < \infty$ such that for each tuple $E = (E_1, \ldots, E_n)$ of measurable subsets of $\mathbb{R}$ and tuple $f = (f_1, \ldots, f_n)$ with $f_j \in X(E_j)$ we have

$$|\Lambda(f)| \leq C|E|^{\alpha}.$$

Restricted type estimates

Definition

A multilinear form Λ is of restricted type $\alpha = (\alpha_1, \ldots, \alpha_n)$ ($\alpha_j \in [0,1]$) if there exists $C < \infty$ such that for each tuple $E = (E_1, \ldots, E_n)$ of measurable subsets of $\mathbb{R}$ and tuple $f = (f_1, \ldots, f_n)$ with $f_j \in X(E_j)$ we have

$$|\Lambda(f)| \leq C|E|^{\alpha}.$$

Theorem [Marcinkiewicz interpolation trick]

Let $\beta = (\beta_1, \ldots, \beta_n)$ be such that $\sum_j \beta_j = 1$ and $\beta_j > 0$ for all j. If Λ is of restricted type α for all α in a neighborhood of β satisfying $\sum_j \alpha_j = 1$, then

$$|\Lambda(f_1, \ldots, f_n)| \leq C \prod_j \|f_j\|_{1/\beta_j}$$

for every f in the linear span of all $X(E)$.

What's left?

We will be done once we show the following: for bounded $G, H \subseteq \mathbb{R}^2$ and $\sum_k |f_k|^2 \leq 1_H$,

$$\left|\left\langle \left(\sum_{|k| \leq k_0} |H_k f_k|^2 \right)^{\frac{1}{2}}, 1_G \right\rangle\right| \lesssim |H|^{\frac{1}{p}} |G|^{1-\frac{1}{p}}. \tag{1}$$

What's left?

We will be done once we show the following: for bounded $G, H \subseteq \mathbb{R}^2$ and $\sum_k |f_k|^2 \leq 1_H$,

$$\left| \left\langle \Big(\sum_{|k| \leq k_0} |H_k f_k|^2 \Big)^{\frac{1}{2}}, 1_G \right\rangle \right| \lesssim |H|^{\frac{1}{p}} |G|^{1-\frac{1}{p}}. \qquad (1)$$

- Restrict attention to $p \in (\frac{3}{2}, 2]$;

What's left?

We will be done once we show the following: for bounded $G, H \subseteq \mathbb{R}^2$ and $\sum_k |f_k|^2 \leq 1_H$,

$$\left|\left\langle \Big(\sum_{|k| \leq k_0} |H_k f_k|^2 \Big)^{\frac{1}{2}}, 1_G \right\rangle\right| \lesssim |H|^{\frac{1}{p}} |G|^{1-\frac{1}{p}}. \tag{1}$$

- Restrict attention to $p \in (\frac{3}{2}, 2]$;
- $|G| \lesssim |H| \Rightarrow p < 2$ ✔;

What's left?

We will be done once we show the following: for bounded $G, H \subseteq \mathbb{R}^2$ and $\sum_k |f_k|^2 \leq 1_H$,

$$\left|\left\langle \Big(\sum_{|k|\leq k_0} |H_k f_k|^2 \Big)^{\frac{1}{2}}, 1_G \right\rangle\right| \lesssim |H|^{\frac{1}{p}} |G|^{1-\frac{1}{p}}. \tag{1}$$

- Restrict attention to $p \in (\frac{3}{2}, 2]$;
- $|G| \lesssim |H| \Rightarrow p < 2$ ✔;
- $|H| \lesssim |G|$: induction on the ratio of $|H|$ and $|G|$.

Key Lemma

Lemma

Let $G', H' \subset [-N, N]^2$ be measurable, and let $\frac{3}{2} < p < 2$. If $|H'| < \frac{1}{10}|G'|$, then there exists a subset $G \subset G'$ (depending only on p, G' and H') with $|G| \geq |G'|/2$ such that (1) holds with $H = H'$, for any sequence of functions f_k with $\sum_k |f_k|^2 \leq 1_H$.

Key Lemma

Lemma

Let $G', H' \subset [-N,N]^2$ be measurable, and let $\frac{3}{2} < p < 2$. If $|H'| < \frac{1}{10}|G'|$, then there exists a subset $G \subset G'$ (depending only on p, G' and H') with $|G| \geq |G'|/2$ such that (1) holds with $H = H'$, for any sequence of functions f_k with $\sum_k |f_k|^2 \leq 1_H$.

On Wednesday:

Key Lemma

Lemma

Let $G', H' \subset [-N,N]^2$ *be measurable, and let* $\frac{3}{2} < p < 2$*. If* $|H'| < \frac{1}{10}|G'|$*, then there exists a subset* $G \subset G'$ *(depending only on* p, G' *and* H'*) with* $|G| \geq |G'|/2$ *such that* (1) *holds with* $H = H'$*, for any sequence of functions* f_k *with* $\sum_k |f_k|^2 \leq 1_H$*.*

On Wednesday:

- Construction of G;

Key Lemma

Lemma

Let $G', H' \subset [-N, N]^2$ be measurable, and let $\frac{3}{2} < p < 2$. If $|H'| < \frac{1}{10}|G'|$, then there exists a subset $G \subset G'$ (depending only on p, G' and H') with $|G| \geq |G'|/2$ such that (1) holds with $H = H'$, for any sequence of functions f_k with $\sum_k |f_k|^2 \leq 1_H$.

On Wednesday:

- Construction of G;
- Size estimate: $|G| \geq |G'|/2$.

Why is this progress?

$$\text{Want to show (1)}: \quad \left|\left\langle \Big(\sum_{|k|\leq k_0} |H_k f_k|^2 \Big)^{\frac{1}{2}}, 1_G \right\rangle\right| \lesssim |H|^{\frac{1}{p}} |G|^{1-\frac{1}{p}}$$

Why is this progress?

Want to show (1) : $$\left|\left\langle\Big(\sum_{|k|\le k_0}|H_k f_k|^2\Big)^{\frac{1}{2}}, 1_G\right\rangle\right| \lesssim |H|^{\frac{1}{p}}|G|^{1-\frac{1}{p}}$$

which follows (CS) from $$\int \sum_{|k|\le k_0}|H_k f_k|^2 1_G \lesssim |H|^{\frac{2}{p}}|G|^{1-\frac{2}{p}}$$

Why is this progress?

Want to show (1) : $$\left|\left\langle\Big(\sum_{|k|\le k_0}|H_k f_k|^2\Big)^{\frac{1}{2}}, 1_G\right\rangle\right| \lesssim |H|^{\frac{1}{p}}|G|^{1-\frac{1}{p}}$$

which follows (CS) from $$\int \sum_{|k|\le k_0}|H_k f_k|^2 1_G \lesssim |H|^{\frac{2}{p}}|G|^{1-\frac{2}{p}}$$

which follows from $$\int \sum_{|k|\le k_0}|H_k f_k|^2 1_G \lesssim \Big(\frac{|G|}{|H|}\Big)^{1-\frac{2}{p}} \int \sum_{k}|f_k|^2$$

Why is this progress?

Want to show (1) : $$\left|\left\langle\left(\sum_{|k|\le k_0}|H_k f_k|^2\right)^{\frac{1}{2}}, 1_G\right\rangle\right| \lesssim |H|^{\frac{1}{p}}|G|^{1-\frac{1}{p}}$$

which follows (CS) from $$\int\sum_{|k|\le k_0}|H_k f_k|^2 1_G \lesssim |H|^{\frac{2}{p}}|G|^{1-\frac{2}{p}}$$

which follows from $$\int\sum_{|k|\le k_0}|H_k f_k|^2 1_G \lesssim \left(\frac{|G|}{|H|}\right)^{1-\frac{2}{p}}\int\sum_k |f_k|^2$$

which follows from $$\|1_G H_k(1_H f)\|_2 \lesssim \left(\frac{|G|}{|H|}\right)^{\frac{1}{2}-\frac{1}{p}}\|f\|_2.$$

ETS $(*)\ \|H_{k,G,H}f\|_2 \lesssim \left(\frac{|G|}{|H|}\right)^{\frac{1}{2}-\frac{1}{p}}\|f\|_2$.

Theorem

Let $\frac{3}{2} < p < 2$ and let $G', H' \subseteq \mathbb{R}^2$ be such that $|H'| < \frac{1}{10}|G'|$. Then there are sets G, H as in the Key Lemma such that for any measurable sets $E, F \subseteq \mathbb{R}^2$ and each $|k| \leq k_0$, we have that

$$(**) \quad |\langle H_{k,G,H}1_F, 1_E\rangle| \lesssim \left(\frac{|G|}{|H|}\right)^{\frac{1}{2}-\frac{1}{p}}|F|^{\frac{1}{2}}|E|^{\frac{1}{2}},$$

ETS $(*)$ $\|H_{k,G,H}f\|_2 \lesssim \left(\frac{|G|}{|H|}\right)^{\frac{1}{2}-\frac{1}{p}}\|f\|_2$.

Theorem

Let $\frac{3}{2} < p < 2$ and let $G', H' \subseteq \mathbb{R}^2$ be such that $|H'| < \frac{1}{10}|G'|$. Then there are sets G, H as in the Key Lemma such that for any measurable sets $E, F \subseteq \mathbb{R}^2$ and each $|k| \le k_0$, we have that

$$(**) \quad |\langle H_{k,G,H}1_F, 1_E\rangle| \lesssim \left(\frac{|G|}{|H|}\right)^{\frac{1}{2}-\frac{1}{p}}|F|^{\frac{1}{2}}|E|^{\frac{1}{2}},$$

Estimate $(*)$ follows from interpolating $(**)$ with

$$|\langle H_{k,G,H}1_F, 1_E\rangle| \lesssim |F|^{\frac{1}{q}}|E|^{1-\frac{1}{q}}, \quad 1 < q < \infty.$$

ETS $(*)$ $\|H_{k,G,H}f\|_2 \lesssim \left(\frac{|G|}{|H|}\right)^{\frac{1}{2}-\frac{1}{p}} \|f\|_2$.

Theorem

Let $\frac{3}{2} < p < 2$ and let $G', H' \subseteq \mathbb{R}^2$ be such that $|H'| < \frac{1}{10}|G'|$. Then there are sets G, H as in the Key Lemma such that for any measurable sets $E, F \subseteq \mathbb{R}^2$ and each $|k| \leq k_0$, we have that

$$(**) \quad |\langle H_{k,G,H} 1_F, 1_E \rangle| \lesssim \left(\frac{|G|}{|H|}\right)^{\frac{1}{2}-\frac{1}{p}} |F|^{\frac{1}{2}} |E|^{\frac{1}{2}},$$

Estimate $(*)$ follows from interpolating $(**)$ with

$$|\langle H_{k,G,H} 1_F, 1_E \rangle| \lesssim |F|^{\frac{1}{q}} |E|^{1-\frac{1}{q}}, \quad 1 < q < \infty.$$

On Wednesday: prove $(**)$.

THANK YOU