1 L^p estimates for the Hilbert transform along a one-variable vector field

after M. Bateman and C. Thiele [2]
A summary written by Diogo Oliveira e Silva

Abstract

The authors of [2] prove L^p estimates for the full Hilbert transform along a measurable, non-vanishing, one-variable vector field in the plane. We summarize their results.

1.1 Introduction

We are interested in singular integral operators on functions of two variables, which act by performing a one-dimensional transform along a particular line in the plane. The choice of lines is to be variable. Thus, for a non-vanishing measurable vector field $v : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{0\}$ and a measurable function $f : \mathbb{R}^2 \to \mathbb{R}$, we define the directional Hilbert transform

$$H_v f(x,y) := p.v. \int_{\mathbb{R}} \frac{f((x,y) - tv(x,y))}{t} dt$$

Specializing to vector fields which depend on one variable only, the authors of [2] are able to prove the following result:

Theorem 1. *Let $p \in (\frac{3}{2}, \infty)$, and let v be a non-vanishing, measurable vector field in the plane such that for all $x, y \in \mathbb{R}$, $v(x,y) = v(x,0)$. Then*

$$\|H_v f\|_p \lesssim \|f\|_p$$

for every $f \in L^p(\mathbb{R}^2)$.

A few remarks may help to further orient the reader:

1. The case of a constant vector field follows from the classical L^p estimates for the one-dimensional Hilbert transform.

2. The class of vector fields depending on the first variable only is invariant under linear transformations which preserve the vertical direction. These symmetries, together with those of the Hilbert transform, allow

us to assume without loss of generality that $v(x,y) = (1, u(x))$ for some real-valued measurable function u satisfying

$$\|u\|_\infty \leq 10^{-2}. \tag{1}$$

3. Sharpness of the endpoint exponent $p = \frac{3}{2}$ is an open problem. It is known however that the exponent in Theorem 1 can be improved to $p = \frac{4}{3}$ under the additional assumption that the function f is an elementary tensor. See [2].

4. The case $p = 2$ of Theorem 1 is actually equivalent to the celebrated Carleson-Hunt theorem on pointwise convergence of Fourier series. See [3].

1.1.1 History of the problem and related work

The discovery of the Besicovitch set in the 1920s inspired Zygmund to ask if integrals of $L^2(\mathbb{R}^2)$ functions could be differentiated in a Lipschitz choice of directions. Much later, Stein raised the singular integral variant of this conjecture: if v is Lipschitz, is (a truncated version of) H_v a bounded operator on $L^2(\mathbb{R}^2)$? For a fuller history of these conjectures, see [3].

In a somewhat different direction, the Hilbert transform along a one variable vector field has been previously studied by Carbery, Seeger, Wainger and Wright, who proved L^p boundedness for $p > 1$ under additional smoothness assumptions on the vector field. On the other hand, Christ, Nagel, Stein and Wainger proved similar estimates under the additional geometric hypothesis that no integral curve of the vector field forms a straight line. For further references, see [2].

Finally, the companion paper [1] proves L^p estimates for $p \in (1, \infty)$ for the Hilbert transform along a one-variable vector field v acting on functions with frequency supported in an annulus. Since their main result will be of importance to us already in the next section, we state it here in a form invariant under the linear transformation group mentioned in remark 2 above:

Theorem 2. *[1] Let $p \in (1, \infty)$, and assume $\widehat{f}(\xi, \eta)$ is supported in a horizontal pair of strips $A < |\eta| < 2A$ for some $A > 0$. Then*

$$\|H_v f\|_p \lesssim \|f\|_p.$$

1.2 The main approach

It is a common theme to reduce L^p estimates for a given operator to restricted weak-type estimates for a model operator. In this spirit, instead of trying to estimate H_v directly, we start by defining the closely related operator

$$H_k := P_k H_v P_c. \quad (k \in \mathbb{Z}/100)$$

Here, by P_c we mean the restriction to a cone in the Fourier plane (ξ, η)

$$\widehat{P_c f}(\xi, \eta) = 1_{\{10|\xi| \le |\eta|\}}(\xi, \eta) \widehat{f}(\xi, \eta),$$

whereas P_k denotes the Fourier multiplier given by $\widehat{P_k f} = 1_{B_k} \widehat{f}$, where B_k is the horizontal pair of bands given by

$$B_k := \{(\xi, \eta) \in \mathbb{R}^2 : |\eta| \in [2^k, 2^{k+.01})\}.$$

By Littlewood-Paley theory and a limiting argument, it will be enough to prove that, for all $k_0 > 0$,

$$\Big\| \Big(\sum_{|k| \le k_0} |H_k f_k|^2 \Big)^{\frac{1}{2}} \Big\|_p \lesssim \Big\| \Big(\sum_{|k| \le k_0} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_p \tag{2}$$

holds[1] for any sequence of functions $f_k \in L^2$, with implicit constant independent of k_0.

Note that H_k is bounded on L^p for $1 < p < \infty$ for each k, by Theorem 2. In particular, (2) is true for $p = 2$. For other values of p, we recall the notion of restricted weak-type estimates in the spirit of ([4], chapter 3) and observe that it suffices to show, for bounded $G, H \subseteq \mathbb{R}^2$ and $\sum_k |f_k|^2 \le 1_H$, that

$$\Big| \Big\langle \Big(\sum_{|k| \le k_0} |H_k f_k|^2 \Big)^{\frac{1}{2}}, 1_G \Big\rangle \Big| \lesssim |H|^{\frac{1}{p}} |G|^{1 - \frac{1}{p}}. \tag{3}$$

In what follows we restrict our attention to the case $\frac{3}{2} < p \le 2$. Since we already have (3) for $p = 2$, we immediately obtain this estimate for $p < 2$ provided $|G| \lesssim |H|$. By a standard inductive procedure, it will suffice to prove the following result:

[1]It is understood that the index k runs through elements of $\mathbb{Z}/100$, that is, multiples of $1/100$.

Lemma 3. *Let $G', H' \subset [-N,N]^2$ be measurable, and let $\frac{3}{2} < p < 2$. If $|H'| < \frac{1}{10}|G'|$, then there exists a subset $G \subset G'$ (depending only on p, G' and H') with $|G| \geq |G'|/2$ such that* (3) *holds with $H = H'$, for any sequence of functions f_k with $\sum_k |f_k|^2 \leq 1_H$.*

In the next section we present the construction of the set G of Lemma 3 and sketch the proof of the size estimate $|G| \geq |G'|/2$. In the last section we outline very briefly how time-frequency analysis comes into play to prove strong L^2 bounds for the sets G and H, from which (3) follows.

1.3 Construction of the set G

Following general principles of wave packet analysis [4], it is natural to decompose the operator H_v into wave packets, which can be visualized by acting with the same group element in the unit square of the plane. The shapes thus obtained are parallelograms with a pair of vertical edges, and because of (1) it is enough to consider parallelograms whose non-vertical edges are close to horizontal.

Given a parallelogram R with two vertical edges, we refer to [2] for the precise definitions of the height $H(R)$, the shadow $I(R)$ and the interval of uncertainty $U(R)$. Given $c > 0$, we denote by cR the parallelogram with the same central line segment as R but height $cH(R)$. We also define

$$E(R) := \{(x,y) \in R : u(x) \in U(R)\}.$$

The following observation will be used several times and its proof is an easy but amusing exercise in elementary geometry which we recommend to the reader:

Lemma 4. *Let R, R' be two parallelograms and assume $R \cap R' \neq \emptyset$, $I(R) = I(R')$, $U(R) \cap U(R') \neq \emptyset$ and $H(R) \leq H(R')$. Then $R \subseteq 7R'$.*

After these preliminaries, we indicate how to construct the set G. Let G' and H' be as in Lemma 3. For $i \in \{1,2\}$, define

$$G_i := \bigcup_{j \in \mathbb{Z}_-} \{R \in \mathcal{R}_i : \frac{|E(R)|}{|R|} \geq 2^j \text{ and } \frac{|H' \cap R|}{|R|} \geq C_\epsilon 2^{-(\frac{1}{2}+\epsilon)j} \Big(\frac{|H'|}{|G'|}\Big)^{\frac{1}{2}}\},$$

where $\mathcal{R}_i$ is a finite set of parallelograms with vertical edges and dyadic shadow adapted to a shifted dyadic grid $\mathcal{I}_i$ on the real line and having some nice properties about which we will not be completely precise. The small parameter $\epsilon > 0$ and the large constant $C_\epsilon < \infty$ will be chosen as a function of p later on in the argument in order to force the set G defined by

$$G' \setminus G := G_1 \cup G_2$$

to satisfy the desired size estimate $|G| \geq |G'|/2$. That this is indeed possible is a consequence of the following result, which holds for parallelograms of *arbitrary* height:

Lemma 5. *Let $\delta, \sigma \in [0,1]$, let H be a measurable set, and let $\mathcal{R}$ be a finite collection of parallelograms with vertical edges and dyadic shadow such that for each $R \in \mathcal{R}$ we have*

$$|E(R)| \geq \delta |R| \text{ and } |H \cap R| \geq \sigma |R|.$$

Then

$$\Big| \bigcup_{R \in \mathcal{R}} R \Big| \lesssim \delta^{-1} \sigma^{-2} |H|.$$

Proof. Adopting the covering lemma approach of Córdoba and Fefferman (see [3]), it will be enough to find a "good" subset $\mathcal{G} \subset \mathcal{R}$ such that

$$\Big| \bigcup_{R \in \mathcal{R}} R \Big| \lesssim \sum_{R \in \mathcal{G}} |R| \quad \text{and} \quad \int (\sum_{R \in \mathcal{G}} 1_R)^2 \lesssim \delta^{-1} \sum_{R \in \mathcal{G}} |R|. \tag{4}$$

We accomplish this by a recursive procedure, which we initialize by setting $\mathcal{G} := \emptyset$ and $STOCK := \mathcal{R}$. As long as we have a nonempty $STOCK$ of parallelograms, we may choose $R \in STOCK$ with maximal $|I(R)|$, and update[2]:

$$\begin{aligned} \mathcal{G} &\leftarrow \mathcal{G} \cup \{R\} \\ \mathcal{B} &\leftarrow \{R' \in STOCK : R' \subset \{x : M_V(\textstyle\sum_{R \in \mathcal{G}} 1_R)(x) \geq 10^{-3}\}\} \\ STOCK &\leftarrow STOCK \setminus \mathcal{B}. \end{aligned}$$

The first inequality in (4) is then a trivial consequence of the Hardy-Littlewood weak-type (1,1) bound. For the second one, we organize the set $\mathcal{P}$ of pairs $(R, R') \in \mathcal{G} \times \mathcal{G}$ such that $R \cap R' \neq \emptyset$ and R is chosen prior to R' into two sets, according to whether the two rectangles are "well-aligned" or not. Define

$$\mathcal{P}' := \{(R, R') \in \mathcal{P} : U(R) \subset 100 U(R')\}$$

[2]Here, M_V denotes the Hardy-Littlewood maximal operator in the vertical direction.

and $\mathcal{P}'' := \mathcal{P} \setminus \mathcal{P}'$. It will be enough to show that, for fixed $R' \in \mathcal{G}$ we have

$$\sum_{R \in \mathcal{R} : (R,R') \in \mathcal{P}'} |R \cap R'| \lesssim |R'|, \tag{5}$$

and that for fixed $R \in \mathcal{G}$ we have

$$\sum_{R' \in \mathcal{R} : (R,R') \in \mathcal{P}''} |R \cap R'| \lesssim \delta^{-1} |R|. \tag{6}$$

The proof of (5) is based on the following observation: if $R' \subset cR$, then $R' \subset \{M_V 1_R > c^{-1}\}$. We use this together with Lemma 4 to show that $H(R) \leq H(R')$ for every $(R, R') \in \mathcal{P}'$, and then the same lemma again to conclude that

$$R \cap (I(R') \times \mathbb{R}) \subset 700R'.$$

It follows that, for some point $(x, y) \in R'$,

$$10^{-3} \geq M_V\Big(\sum_{R:(R,R') \in \mathcal{P}'} 1_R\Big)(x, y) \geq \frac{1}{700} \sum_{R:(R,R') \in \mathcal{P}'} \frac{|R \cap R'|}{|R'|}.$$

We omit the proof of (6), hoping to say something about it at the Summer School. □

1.4 The "end" of the proof

In what follows, we omit almost all details.

Let $p \in (\frac{3}{2}, 2)$, and let $G', H' \subseteq \mathbb{R}^2$ be as in Lemma 3. Once again by restricted weak-type interpolation, it will be enough to establish the following single frequency band estimate: for any measurable sets $E, F \subseteq \mathbb{R}^2$ and each $|k| \leq k_0$, we have that

$$|\langle H_{k,G,H} 1_F, 1_E \rangle| \lesssim \left(\frac{|G|}{|H|} \right)^{\frac{1}{2} - \frac{1}{p}} |F|^{\frac{1}{2}} |E|^{\frac{1}{2}}, \tag{7}$$

where $G \subset G'$ is the set constructed in the last section, $H := H'$, and the operator $H_{k,G,H}$ is defined by

$$H_{k,G,H} f := 1_G H_k (1_H f).$$

Assuming without loss of generality that $E \subset G$ and $F \subset H$, we have that $\langle H_{k,G,H} 1_F, 1_E \rangle = \langle H_k 1_F, 1_E \rangle$. Following [1], we write the latter form as a linear combination of a bounded number of model forms

$$\langle H_k 1_F, 1_E \rangle = \sum_{s \in \mathcal{U}_k} \langle C_{s,k} 1_F, 1_E \rangle, \tag{8}$$

where $\mathcal{U}_k$ is a set of parallelograms with vertical edges and height depending on k only. To estimate the sum in (8), one starts by proving estimates for the sum over certain subsets of $\mathcal{U}_k$ called trees. Each tree T is assigned a parallelogram $\mathrm{top}(T)$, a density $\delta(T)$ and a size $\sigma(T)$. One obtains for each tree T:

$$\sum_{s \in T} |\langle C_s 1_F, 1_E \rangle| \lesssim \delta\sigma |\mathrm{top}(T)|.$$

Denoting by $\mathcal{T}_{\delta,\sigma}$ the collection of trees with density at most δ and size at most σ, it remains to estimate $\sum_{\delta,\sigma} S_{\delta,\sigma}$ with

$$S_{\delta,\sigma} := \sum_{T \in \mathcal{T}_{\delta,\sigma}} \delta\sigma |\mathrm{top}(T)|.$$

The desired estimate (7) follows from the estimates for $S_{\delta,\sigma}$ proved in [1] (and presented in J. Jung's summary) together with one new maximal estimate.

I am indebted to Michael Bateman for a very useful discussion of some parts of [1] and [2].

References

[1] Bateman, M., *Single Annulus L^p estimates for Hilbert transforms along vector fields.* Preprint (2011).

[2] Bateman, M. and Thiele, C., *L^p estimates for the Hilbert transform along a one-variable vector field.* Preprint (2011).

[3] Lacey, M. and Li, X., *On a Conjecture of E.M. Stein of the Hilbert Transform on Vector Fields*, Memoirs of the AMS **205** (2010), no. 965.

[4] C. Thiele, *Wave Packet Analysis,* CBMS Regional Conference Series in Mathematics, 105. American Mathematical Society, Providence, RI, 2006.

Diogo Oliveira e Silva, UC Berkeley
email: **dosilva@math.berkeley.edu**