MATH 55 WORKSHEET 1: PROOF TECHNIQUES (Solutions)

1 Let $L(x,y)$ = "x loves y." Express the statement: "Everybody loves my baby but my baby doesn't love anybody but me" with predicates and quantifiers. Who is "my baby"?

One example: $\forall x\,(L(x, \text{my baby}) \wedge (x = \text{me} \longleftrightarrow L(\text{my baby}, x)))$

By universal instantiation, $L(\text{my baby}, \text{my baby})$. Then by our biconditional, $L(\text{my baby}, \text{my baby}) \longrightarrow \text{my baby} = \text{me}$.

2 Prove that the sum of an irrational number and a rational number is irrational.

By contradiction: suppose a is rational, c is irrational, but $a+c$ is rational. Then we can say $a = \frac{m}{n}$ and $a+c = \frac{p}{q}$ for integers m, n, p, q. So $c = \frac{p}{q} - a = \frac{p}{q} - \frac{m}{n} = \frac{np - qm}{qn}$ which is rational.

contradiction!

3 Show that the product of two rational numbers is rational.

By direct proof: $\frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq} = \frac{\text{integer}}{\text{integer}} = \text{rational}$.

4 Prove that if n is an integer and $3n+2$ is even, then n is even.

By contradiction: [Suppose $3n+2$ even] but n odd. So $n = 2k+1$. Then $3n+2 = 3(2k+1)+2 = 6k+5$ which is odd. [contradiction!]

By contrapositive: remove bracketed sentences to get a proof by contrapositive.

5 Show that if a, b, and c are real numbers and $a \neq 0$, then there is a unique solution to the equation $ax + b = c$.

$\exists$: $x = \frac{c-b}{a}$ is a solution.

$!$: suppose y is a solution too. Then $ax+b = c = ay+b$. So $ax = ay$. So $x = y$.

6 Prove that $2^n > n^2$ for $n > 4$. (note: this is a hard one).

Induction proof:

(1) Base step: $n=5$: $2^5 = 32$ $5^2 = 25$, $32 > 25$. ✓

(2) Inductive step: Assume $2^k > k^2$, and let's show $2^{k+1} > (k+1)^2$.

Observe: $2^{k+1} = 2 \cdot 2^k > 2 \cdot k^2 = k^2 + k^2 > k^2 + 4k \geq k^2 + 2k + 1$
$= (k+1)^2$.

(because $k > 4$)