Math 55 Midterm 2 review solutions

1 (a) $\underline{10}\cdot\underline{9}\cdot\underline{8}\cdot\underline{7}\cdot\underline{6}\cdot\underline{5} = \boxed{P(10,6)}$

(b) $\underline{10}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10} = \boxed{10^6}$

(c) $\underline{10}\cdot\underline{9}\cdot\underline{8}\cdot\underline{7}\cdot\underline{6}\cdot\underline{5} \big/ 6! = \boxed{\binom{10}{6}}$

(d) Balls and walls: 6 indistinguishable balls and $10-1=9$ walls: $\boxed{\binom{15}{6} \text{ or } \binom{15}{9}.}$

2 $=$ # start with 000 + # end with 1111 − # that do both

$= 2^7 + 2^6 - 2^3 = \boxed{184}$

3 (a) $= 999 -$ # with no 9's $= 999 - (\underline{9}\cdot\underline{9}\cdot\underline{9} - 1) = \boxed{271}$

000 isn't included.

(b) $= \underline{5}\cdot\underline{5}\cdot\underline{5} - 1 = \boxed{124}$

(c) $=$ #55□ + #□55 − #555 $= 10 + 10 - 1 = \boxed{19}$

(d) $=$ # 1-digit palindromes + # 2-digit palindromes + # 3-digit palindromes

$= 9 + 9 + \underline{9}\cdot\underline{10}\cdot\underline{1}$

$= \boxed{108}$

4 There are $\binom{10}{5} = 252$ total subsets of 5 elements

The biggest the sum can be is $50+49+48+47+46 = 240$

And the smallest it can be is $1+2+3+4+5 = 15.$

So there are 226 total possible sums. By pigeonhole principle some subsets must have equal sums.

5 By PHP, two of the numbers must be consecutive. Consecutive integers are coprime.

6 (a) $\boxed{n=5}$ (b) $\boxed{n=7}$ (c) $\dfrac{n!}{3!(n-3)!} = \dfrac{n!}{n-2!} \longrightarrow 3!(n-3)! = (n-2)!$

$\longrightarrow 6 = n-2 \longrightarrow \boxed{n=8}$

7 Solution 1: no pairs means 1 of every rank; with 4 possible suits each:

$$4^{13} \Big/ \binom{52}{13}$$

Solution 2: Pick one card from each rank: $\binom{52}{1}\binom{48}{1}\binom{44}{1}\cdots\binom{4}{1}\cdot\frac{1}{13!} \Big/ \binom{52}{13}$

Then remove the order by dividing by 13!

(These two answers are the same!)

8 $4!$ ways to deal the aces. $\binom{48}{12}\binom{36}{12}\binom{24}{12}\binom{12}{12} = \frac{48!}{12!12!12!12!}$ ways to deal the rest. $\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13} = \frac{52!}{13!13!13!13!}$ ways to deal in total.

$$\hookrightarrow \quad \frac{4!\,48!}{12!\,12!\,12!\,12!} \Big/ \frac{52!}{13!13!13!13!} = \frac{4!\,13^4}{52\cdot 51\cdot 50\cdot 49} = \boxed{\frac{13^3}{17\cdot 25\cdot 49}}$$

9 Multiples of p less than pq: $p, 2p, 3p, \dots, (q-1)p$

Multiples of q less than pq: $q, 2q, 3q, \dots, (p-1)q$

So there are $q-1+p-1 = p+q-2$ numbers that work.

So, probability $= \boxed{\frac{p+q-2}{pq-1}}$

10 If n is even: $\frac{2^{n/2}}{2^n} = \boxed{\frac{1}{2^{n/2}}}$.

If n is odd: $\frac{2^{\frac{n+1}{2}}}{2^n} = \boxed{\frac{1}{2^{\frac{n-1}{2}}}}$.

$= \boxed{\frac{1}{2^{\lfloor \frac{n}{2} \rfloor}}}$

11 Balls + walls: $26-1=25$ walls separating buckets for each letter A–Z, 5 balls. e.g.: $o|oo|||o|\cdots|o|$ = "ABBEY."

A B C D E Y Z

So $\boxed{\binom{30}{25} \text{ or } \binom{30}{5}}$

12 $V(\text{uniform distribution}) = \frac{n^2-1}{12}$. $V(\text{6-sided}) = \frac{35}{12}$, $V(\text{8-sided}) = \frac{63}{12}$.

Since the rolls are <u>independent</u>, $V(\text{sum}) = \frac{35}{12} + \frac{63}{12} = \frac{98}{12} = \boxed{\frac{49}{6}}$

13

$$(3n+2) = (2n+1)\cdot 1 + (n+1)$$
$$(2n+1) = (n+1)\cdot 1 + (n)$$
$$(n+1) = (n)\cdot 1 + \boxed{1}$$
$$(n) = 1\cdot n + 0$$

$\boxed{14}$ If $n = d_0 + 10d_1 + 100d_2 + \cdots + 10^k d_k$ then

$$9 \mid n \text{ iff } 0 \equiv d_0 + 10d_1 + 100d_2 + \cdots + 10^k d_k$$
$$\equiv d_0 + d_1 + d_2 + \cdots + d_k$$
$$= \text{sum of digits} \pmod 9$$
$$\text{iff } 9 \mid \text{sum of digits.}$$

$\left\{ \begin{array}{l} \text{we used the fact that} \\ 10 \equiv 1 \pmod 9 \end{array} \right\}$

$\boxed{15}$ By a similar argument, observe that $10 \equiv (-1) \pmod{11}$,

so
$$11 \mid n \text{ iff } 0 \equiv d_0 + 10d_1 + 100d_2 + \cdots + 10^k d_k$$
$$= d_0 - d_1 + d_2 + \cdots + (-1)^k d_k$$
$$= \text{alternating sum of digits} \pmod{11}$$
$$\text{iff } 11 \mid \text{alternating sum of digits.}$$

$\boxed{16}$ Divisors of n: $1, n$, other divisors.

$$\text{Sum of divisors} = 1 + n + \text{other divisors}$$
$$= 1 + n \text{ iff there are no other divisors}$$
$$\text{iff } n \text{ is prime.}$$

$\boxed{17}$ $ABCABC = 1000 \cdot (ABC) + (ABC) = 1001 \cdot (ABC) = 7 \cdot 11 \cdot 13 (ABC)$.

$\boxed{18}$ Factors always come in pairs, so a number will have an even number of factors <u>unless</u> one factor is paired with itself. This describes the set of perfect squares.

So
$$1 + 4 + 9 + \cdots + 100 = \sum_{k=1}^{n=10} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{10 \cdot 11 \cdot 21}{6}$$
$$= \boxed{385}$$

$\boxed{19}$ $X \equiv 4 \pmod 6 \longrightarrow X \equiv 0 \pmod 2$

$\searrow X \equiv 1 \pmod 3$

$X \equiv 13 \pmod{15} \longrightarrow X \equiv 3 \pmod 5.$

Since the two congruences agree with what X should be congruent to mod 3, we can use CRT!

CRT: $X \equiv 0 \cdot (3\cdot5)\overline{(3\cdot5)} + 1\cdot(2\cdot5)\overline{(2\cdot5)} + 3\cdot(2\cdot3)\overline{(2\cdot3)}$

$= 0 + 1\cdot 10 \cdot 1 + 3 \cdot 6 \cdot 1$

$= 28 \pmod{30}.$

So all solutions are $\boxed{X = 28 + 30n}$

$\boxed{20}$ $X \equiv 2 \pmod 6 \longrightarrow X \equiv 2 \pmod 3$

$X \equiv 3 \pmod 9 \longrightarrow X \equiv 0 \pmod 3$

Since the two congruences disagree on what x is congruent to mod 3, we cannot solve this system!