MATH 55 MIDTERM 2

Summer Sessions 2007 J. Doker

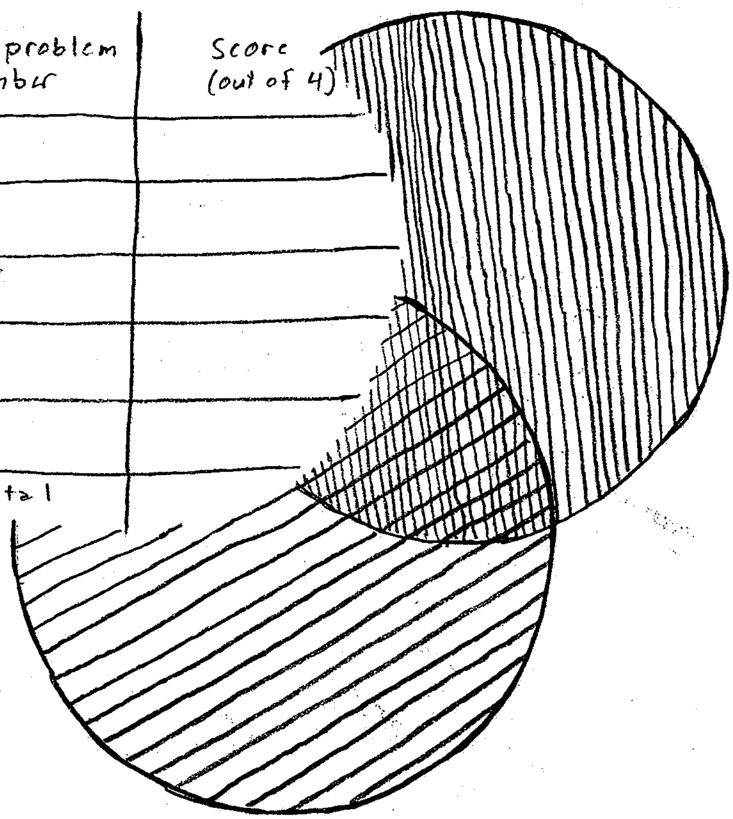
Name: Solutions

Grading: I graded your 4 best problems.

Mean = 11.4/16 stdev = 4.

Curve:	F	D	C	B	A
	0 – 5	6 – 7	8 – 10	11 – 13	14 – 16

problem number	Score (out of 4)
1	
2	
3	
4	
5	
Total	

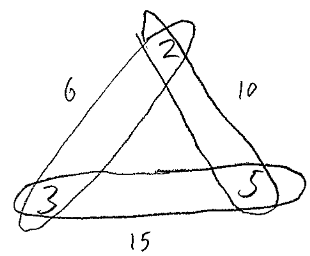


1 A set of integers is called <u>mutually relatively prime</u> if the greatest common divisor of these integers is 1. Find a set of three mutually relatively prime integers such that no two of them are relatively prime.

(4pts)

If p, q, r are prime, then pq, qr, and pr will suffice.

e.g. $p=2$, $q=3$, $r=5$ $\longrightarrow$ 6, 10, 15 works.



2 Find the probability that an anagram of the word PEPPERCORN starts or ends with P. Is it more than $\frac{1}{2}$?

(4pts)

\# That start or end in P = # start with P + # end with P − # both

$$= \frac{9!}{2!2!2!} + \frac{9!}{2!2!2!} - \frac{8!}{2!2!} = N.$$

$$\# \text{ Total anagrams} = \frac{10!}{3!2!2!}.$$

$$P(\text{start or end in P}) = \frac{N}{10!/3!2!2!} = \boxed{8/15} > \frac{1}{2}.$$

$\boxed{3}$ (a) The fact that if you pick $n+1$ odd positive integers less than or equal to $2n$ then you must pick some integer more than once is an application of the <u>pigeon hole</u> principle.

(2 pts) The fact that any positive integer can be written uniquely as the product of an odd number and a power of 2 is an application of the fundamental theorem of <u>Arithmetic</u>.

(b) Use the above two facts to prove the following:
If you pick $n+1$ unique positive integers less than or equal to $2n$ then one of them must divide another.

(2 pts)

- We have $n+1$ integers
- Therefore we have $n+1$ odd parts (Fact 2)
- Therefore 2 odd parts must be the same. (Fact 1)
- Therefore 2 of our numbers are $2^m \cdot N$, $2^k \cdot N$, where $m > k$ and N is the common odd part.
- $2^k \cdot N \mid 2^m \cdot N$ because $\frac{2^m \cdot N}{2^k \cdot N} = 2^{m-k}$, an integer.

4 (a) Let X be a random variable and a be a real number. Prove that $V(aX) = a^2 V(X)$.

(2 pts)

$$\boxed{\begin{aligned} V(aX) &= E((aX)^2) - E(aX)^2 \\ &= E(a^2X^2) - (a\,E(X))^2 \\ &= a^2 E(X^2) - a^2 E(X)^2 \\ &= a^2 (E(X^2) - E(X)^2) \\ &= a^2 V(X). \end{aligned}}$$

(b) Let X be a random variable and b be a real number. Prove that $V(X+b) = V(X)$.

(2 pts)

b is constant, so X and b are independent.

Hence $V(X+b) = V(X) + V(b) = V(X) + 0$.

You can also prove this algebraically like in (a).

5 (a) Prove that if n is an odd integer greater than 1 then n and $n-2$ are relatively prime.

Suppose $d \mid n$ and $d \mid n-2$.

Then $d \mid (n-(n-2)) \longrightarrow d \mid 2$.

$d \neq 2$ because n and $n-2$ are odd.

So $d=1$. Hence the only common divisor is 1.

(1 pt)

— Or —

Euc. Alg: Let $n = 2k+1$, $n-2 = 2k-1$:

$$2k+1 = (2k-1)1 + 2$$
$$2k-1 = 2(k-1) + 1$$
$$2 = 1(2) + 0$$

(b) Solve the following linear congruences using any method of your choice: $3w \equiv 1 \pmod 5$, $5x \equiv 1 \pmod 7$, $7y \equiv 1 \pmod 9$, $9z \equiv 1 \pmod{11}$.

(1 pt)

$$w \equiv 2 \pmod 5$$
$$x \equiv 3 \pmod 7$$
$$y \equiv 4 \pmod 9$$
$$z \equiv 5 \pmod{11}$$

{part (c) on back}

5(c) Find a formula for the inverse of $n-2$, modulo n, when n is an odd integer greater than 1. Prove your formula is correct.

By part (b) we see a pattern:

n	$n-2$	$\overline{(n-2)} \pmod{n}$
5	3	2
7	5	3
9	7	4
11	9	5

(2 pts)

So we propose that $\overline{(n-2)} \equiv \frac{(n-1)}{2} \pmod{n}$.

Indeed: $(n-2) \cdot \frac{(n-1)}{2} \equiv (-2) \cdot \frac{(n-1)}{2}$

$= -n+1$

$\equiv 1 \pmod{n}$.