Wednesday, 8/15/2007

Planar graphs and how we color them

Last time we learned some facts about connected, simple, planar graphs with V = vertex set, E = edge set, R = region set.

- $\sum_{v \in V} \deg(v) = 2|E|$
- $\sum_{r \in R} \deg(r) = 2|E|$ (Think of $\deg(r)$ = # of walls you have to paint in room r.)
- $|V| - |E| + |R| = 2$ (Euler's theorem).

We can use these facts in several ways:

- <u>Corollary 1</u>: If G is connected, simple, planar with $|V| \geq 3$, then
$$|E| \leq 3|V| - 6.$$

Notice that G is <u>simple</u>, so there's no loops or double edges. Hence, every region has degree at least 3. So

$$2|E| = \sum_{r \in R} \deg(r) \geq \sum_{r \in R} 3 = 3|R| = 3[2 - |V| + |E|]$$
$$= 6 - 3|V| + 3|E|.$$

$\longrightarrow$ $|E| \leq 3|V| - 6$. ✓

We can use this to prove K_5 is nonplanar! Indeed, if it were, then $10 = |E| \leq 3|V| - 6 = 3 \cdot 5 - 6 = 9$. Contradiction!

- <u>Corollary 2</u>: If, in addition, G has no triangles (cycles of length 3) then $|E| \leq 2|V| - 4$.

This is similar to Cor-1 except now $\deg(r) \geq 4$ $\forall r \in R$. So
$$2|E| \geq 4|R| = 4[2 - |V| + |E|] = 8 - 4|V| + 4|E|.$$
$\longrightarrow$ $2|E| \leq 4|V| - 8$ $\longrightarrow$ $|E| \leq 2|V| - 4$.

We can use this to prove that $K_{3,3}$ is nonplanar, since $K_{3,3}$ contains no cycles of odd length (why?). Indeed, if $K_{3,3}$ were planar then $9 = |E| \leq 2|V| - 4 = 2 \cdot 6 - 4 = 8$. Contradiction!

- **Corollary 3:** Any connected simple planar graph must have some vertex of degree ≤ 5.

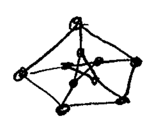
If not, then $\deg(v) \geq 6 \ \forall v \in V$. Hence

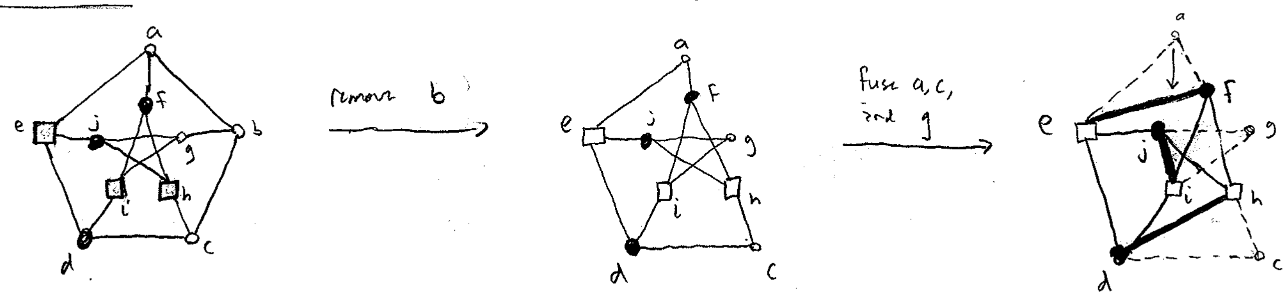
$$6|V| - 12 \geq 2|E| = \sum_{v \in V} \deg(v) \geq 6|V|,$$ a contradiction!

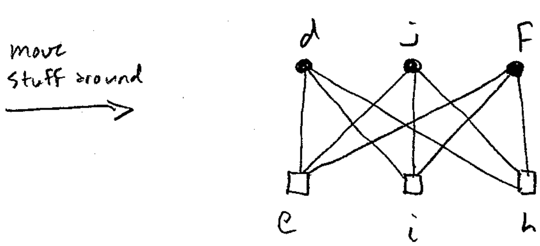
$\uparrow$ (Corollary 1)

- **Kuratowski's Theorem:** A graph is nonplanar iff it contains a subgraph homeomorphic to K_5 or $K_{3,3}$.

— homeomorphic means a subgraph where you're allowed to fuse edges, like this:

— Eg: The Peterson graph  is non planar.





$= K_{3,3}$!

{We ended with a proof of the 5-color theorem, which says you can color any normal map with only 5 colors so that no 2 countries of the same color are touching. — My favorite proof in all of mathematics! Adios folks.
We'll let that be a treat for those who come to class.
— J. D. 2007}