6/27/2007 Wednesday

## Section 1.3: Predicates and Quantifiers

Sometimes we have a propositon whose truth value may change based on a variable in the proposition. For example, the proposition "X is a math major" may be true if X = "Jeff Doker" but false if X = "Lance Armstrong." Propositions like these are called <u>propositional functions</u>, and we denote them like $P(x)$, for instance. So $P$(Jeff Doker) = "Jeff Doker is a math major."

Just like in English, the x part is called the subject and the "is a math major" part is called the <u>predicate</u>.

Sometimes we may want to know the truth values for a propositional function over a bunch of different values of x. We have things called <u>quantifiers</u> to help us do this: the phrase "For all" is called the <u>universal quantifier</u> and is denoted by "$\forall$." The phrase "there exists" is called the <u>existential quantifier</u> and is denoted by "$\exists$." Here's how they work:

Let's say we're looking at the set of people in our class. Then:

"$\underbrace{\boxed{\text{For all}}}_{\forall}$ $\underbrace{\boxed{\text{x in our class}}}_{x}$, $\underbrace{\boxed{\text{x is a math major}}}_{P(x)}$." can be written: $\forall x P(x)$.

Also,

"$\underbrace{\boxed{\text{There exists}}}_{\exists}$ an $\underbrace{\boxed{\text{x in our class}}}_{x}$ such that $\underbrace{\boxed{\text{x is a math major}}}_{P(x)}$."

can be written: $\exists x P(x)$.

In the case of our class, $\forall x P(x)$ is false and $\exists x P(x)$ is true. But if we were instead looking at the set of people who share an office with me in Evans, $\forall x P(x)$ would be a true statement. The set that we take our x values from is called the <u>domain</u>.

Here's a summary of these quantifiers:

| Statement | When true? | When false? |
|---|---|---|
| $\forall x\, P(x)$ | $P(x)$ is true for every $x$. | There is an $x$ for which $P(x)$ is false |
| $\exists x\, P(x)$ | There is an $x$ for which $P(x)$ is true | $P(x)$ is false for every $x$. |

From this we can reason that the negations of these quantifiers look like this:

$$\boxed{\begin{aligned} \neg \forall x\, P(x) &\equiv \exists x\, \neg P(x) \\ \neg \exists x\, P(x) &\equiv \forall x\, \neg P(x) \end{aligned}}$$

Quantifiers have higher precedence than all logical operators from yesterday. For example, $\forall x\, P(x) \wedge Q(x) \equiv \left(\forall x\, P(x)\right) \wedge Q(x)$.

Determining whether two statements involving predicates and quantifiers are logically equivalent is a bit different than before, because we have to take into account all possible values of the domain. A good example of this is example 19 on page 39, which shows how to prove $\forall x\,(P(x) \wedge Q(x))$ is logically equivalent to $\forall x\, P(x) \wedge \forall x\, Q(x)$. We will work out other examples in class, and there are a lot of good ones in the text that you should peruse too.

## Section 1.4: Nested Quantifiers

Sometimes we want to use a propositional function that involves more than one variable. For instance let $P(x,y) =$ "$x$ loves $y$."

Let's look at the following statements, with domain being our class:

"$\underbrace{\boxed{\text{For every } x}}_{\forall x}$ $\underbrace{\boxed{\text{and } y}}_{\forall y}$ in our class, $\underbrace{\boxed{x \text{ loves } y}}_{P(x,y)}$." $\equiv$ Everybody loves everybody.

"$\underbrace{\boxed{\text{There is some } x}}_{\exists x}$ and $\underbrace{\boxed{\text{some } y}}_{\exists y}$ such that $\underbrace{\boxed{x \text{ loves } y}}_{P(x,y)}$." $\equiv$ Somebody loves somebody.

"$\underbrace{\boxed{\text{There is some } x}}_{\exists x}$ such that $\underbrace{\boxed{\text{for all } y}}_{\forall y}$ $\underbrace{\boxed{x \text{ loves } y}}_{P(x,y)}$." $\equiv$ somebody loves everybody.

"$\underbrace{\boxed{\text{For every } x}}_{\forall x}$ $\underbrace{\boxed{\text{there is a } y}}_{\exists y}$ such that $\underbrace{\boxed{x \text{ loves } y}}_{P(x,y)}$." $\equiv$ everybody loves somebody.

How touching.

Note that we can negate nested quantifiers using the rules for negating single quantifiers:

$$\boxed{\begin{array}{l} \neg \exists x \forall y \ P(x,y) \equiv \forall x \exists y \neg P(x,y) \\ \neg \forall x \exists y \ P(x,y) \equiv \exists x \forall y \neg P(x,y) \end{array}}$$

You should verify that these make sense based on the previous example for $P(x,y)$.