Tuesday
8/7/2007

Commonly-used Generating functions

(1) $\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}$ finite geometric

(2) $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ infinite geometric

(5) extended binomial thm (see next pg)

(3) $\sum_{k=0}^{\infty} (k+1)x^k = \frac{1}{(1-x)^2}$

(4) $\sum_{k}^{n} \binom{n}{k} x^k = (1+x)^n$ binomial

~~If n isn't a positive integer, the sequence~~ will be the sequence of extended binomial coefficients

Def The extended binomial coefficient $\binom{r}{k}$ is defined as

$$\binom{r}{k} = \begin{cases} r(r-1)\cdots(r-k+1)/k! & \text{if } k>0 \\ 1 & \text{if } k=0 \end{cases}$$

Pretty similar to our standard binomial coefficient!

$$\binom{r}{k} = \frac{r!}{k!(r-k)!} = \frac{r\cdot(r-1)\cdots(r-k+1)}{k!}$$

Exs of Extended binomial coefficient

$$\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3\cdot 2\cdot 1} = -4$$

$$\binom{1/2}{3} = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\cdot 2\cdot 1} = \frac{1}{16}$$

w/ n a positive integer

$$\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} = (-1)^k \frac{(n)(n+1)(n+2)\cdots(n+k-1)}{k!}$$

$$= \binom{n+k-1}{k}(-1)^k \leftarrow \text{like balls + walls!}$$

So, the $\boxed{\text{Extended binomial thm}}$:

$$\sum_{k=0}^{\infty} \binom{r}{k} x^k = (1+x)^r \quad (\text{where } r \in \mathbb{R})$$

Note: in our old binomial thm we only summed $\sum_{k=0}^{n}$, b/c $\binom{n}{k}=0$ if $k>n$, but if n is not a positive integer then we don't have that problem, so in E.B.T. we sum to ∞

Ex
$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k$$
$$= \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k$$

Ex
$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k$$
$$= \sum_{k=0}^{\infty} \cancel{(-1)^k} \binom{n+k-1}{k} \cancel{(-1)^k} \cdot x^k$$
$$= \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

cool generating function for balls + walls $\frac{1}{(1-x)^n}$

→ So the # of ways to place k balls around $n-1$ walls is equal to the coefficient of x^k in the power series expansion of $\frac{1}{(1-x)^n}$

Pg 489 has a bunch of cool generating functions
like power series for e^x and $\ln(1+x)$

Uses of generating functions

- counting problems

Ex How many integer solutions are there to this eqtn:

$a+b+c=17$ when $2 \le a \le 5$, $3 \le b \le 6$ and c is even

A: $G(x) = (x^2+x^3+x^4+x^5)(x^3+x^4+x^5+x^6)(1+x^2+\ldots+x^{2k}+\ldots)$

exponents correspond to possibilities for a, b, c

The answer is the coefficient of x^{17} in the expansion of the polynomial $G(x)$. Still have to count, and you get $\boxed{8}$

Let's try rewriting that $G(x)$ from above.

$$(x^2+x^3+x^4+x^5) = x^2(1+x+x^2+x^3) = x^2 \sum_{k=0}^{3} x^k = x^2\left(\frac{x^4-1}{x-1}\right)$$

$$(x^3+x^4+x^5+x^6) = x^3(1+x+x^2+x^3) = x^3\left(\frac{x^4-1}{x-1}\right)$$

$$(1+x^2+x^4+x^6+\ldots) = \sum_{k=0}^{\infty} x^{2k} = \sum (x^2)^k = \frac{1}{1-x^2}$$

So $\boxed{G(x) = x^5\left(\frac{x^4-1}{x-1}\right)^2\left(\frac{1}{1-x^2}\right)}$

Ex How many ways can you make change for \$1

$$G(x) = (\underbrace{1+x+x^2+\ldots}_{\text{pennies}})(\underbrace{1+x^5+x^{10}+\ldots}_{\text{nickels}})(\underbrace{1+x^{10}+x^{20}+\ldots}_{\text{dimes}})(\underbrace{1+x^{25}+x^{50}+\ldots}_{\text{quarters}})(\underbrace{1+x^{50}+\ldots}_{\text{fiftycent pc}})$$

The answer is the coefficent of x^{100}

$G(x) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^{10}}\right)\left(\frac{1}{1-x^{25}}\right)\left(\frac{1}{1-x^{50}}\right)$ simplified

Next time:
Use gen functions to solve recurrence relations,
prove ~~combinatorial~~ strange identities