NAME: Solutions!

MATH 54

MIDTERM TWO

1	5
2	5
3	6
4	5
5	4
6	5
7	5
8	4
9	5
B	4

Total: 48

Instructor: J. Doker

1 Find the matrix representation for the linear transformation that maps

$$\begin{bmatrix}1\\3\end{bmatrix} \longmapsto \begin{bmatrix}-1\\-1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}2\\7\end{bmatrix} \longmapsto \begin{bmatrix}2\\1\end{bmatrix}.$$

Let $A = \begin{bmatrix}1 & 2\\3 & 7\end{bmatrix}$ and $B = \begin{bmatrix}-1 & 2\\-1 & 1\end{bmatrix}$.

A maps $e_1 \mapsto \begin{bmatrix}1\\3\end{bmatrix}$, $e_2 \mapsto \begin{bmatrix}2\\7\end{bmatrix}$ and B maps $e_1 \mapsto \begin{bmatrix}-1\\-1\end{bmatrix}$, $e_2 \mapsto \begin{bmatrix}2\\1\end{bmatrix}$.

So BA^{-1} maps $\begin{bmatrix}1\\3\end{bmatrix} \mapsto e_1 \mapsto \begin{bmatrix}-1\\-1\end{bmatrix}$

$\begin{bmatrix}2\\7\end{bmatrix} \mapsto e_2 \mapsto \begin{bmatrix}2\\1\end{bmatrix}$.

$$BA^{-1} = \begin{bmatrix}-1 & 2\\-1 & 1\end{bmatrix}\begin{bmatrix}7 & -2\\-3 & 1\end{bmatrix} = \boxed{\begin{bmatrix}-13 & 4\\-10 & 3\end{bmatrix}}.$$

5 pts

2 Find the distance from the point $(4, 2)$ to the line $y = 2x + 7$.

First subtract 7 from the y-coord of everything to get a setup where the line hits the origin:

Let $u = \begin{pmatrix}1\\2\end{pmatrix}$ be a basis for the line. Then if $v = \begin{pmatrix}4\\-5\end{pmatrix}$,

$$\hat{v} = \text{proj}_u v = \frac{-6}{5}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-6/5\\-12/5\end{pmatrix};$$

and the distance is $\|v - \hat{v}\| = \left\|\begin{pmatrix}26/5\\-13/5\end{pmatrix}\right\| = \sqrt{\frac{26^2}{25} + \frac{13^2}{25}}$

$$= \sqrt{\frac{13^2}{25}(2^2+1)} = \boxed{\frac{13\sqrt{5}}{5}}.$$

5pts

3 Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. A has eigenvalues 1 and 5.

(a) If possible, find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

$\lambda = 1$: $A - I = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad E_1 = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$\lambda = 5$: $A - 5I = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & -4 \\ 0 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad E_5 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$S = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

4 pts

(b) If possible, find a diagonal matrix B such that $A^3 - 2A^2 + I = SBS^{-1}$ (where S is as above).

$$\begin{aligned} A^3 - 2A^2 + I &= SD^3S^{-1} - 2SD^2S^{-1} + SIS^{-1} \\ &= S(D^3 - 2D^2 + I)S^{-1} \\ &= S \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 76 \end{bmatrix} S^{-1}. \end{aligned}$$

2 pts

4. Write out the spectral decomposition of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.
(i.e. write A as a linear combination of 2×2 matrices weighted by the eigenvalues of A.)

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 9 - 6\lambda + \lambda^2 - 1$$
$$= \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2).$$

$\lambda = 4$: $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. $E_4 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right\}$.

$\lambda = 2$: $A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. $E_2 = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right\}$.

$$A = SDS^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= 4 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} + 2 \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= 4 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + 2 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

5pts

5. Fill in the blanks in this proof of the triangle inequality: 4pts

$$\|u+v\|^2 \leq \|u\|^2 + 2|\langle u, v\rangle| + \|v\|^2 \leq \boxed{\|u\|^2 + 2\|u\|\|v\| + \|v\|^2} \leq \boxed{(\|u\| + \|v\|)^2}.$$

(Remember, the triangle inequality states that the length of the sum of two vectors is at most the sum of the two vectors' lengths.)

The first blank is due to the Cauchy-Schwarz inequality

6 Consider the data set $(2,5)$ and $(-2,3)$. If you wanted to find a best-fit parabola $y = at^2 + bt + c$ for this data, you would need to solve the least-squares problem $A^TAx = A^Ty$.

(a) Find A for this data set.

$(2,5)$: $4a + 2b + c = 5$

$(-2,3)$: $4a - 2b + c = 3$

$$\begin{bmatrix} 4 & 2 & 1 \\ 4 & -2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 4 & -2 & 1 \end{bmatrix}.$$

3 pts

(b) Does this problem have a <u>unique</u> least squares solution? Why/why not?

No — the columns of A are linearly dependent, so A^TA will not be invertible and so we'll have free variables.

2 pts

7 The orthogonal complement of the line spanned by $\begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}$ is a plane whose equation is $ax_1 + bx_2 + cx_3 = 0$.
Find a, b, and c. (Hint: use the definition of "orthogonal.")

The ortho. complement of $\text{Span}\begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$ is the set of all vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ who are orthogonal to $\begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$, i.e. whose dot product with $\begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$ is zero, i.e. $\begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 6x_1 - x_2 + 2x_3 = 0.$

Well look at that — we've found the equation for our plane.

$a = 6 \quad b = -1 \quad c = 2.$

5 pts

8. Prove that if U is a square orthogonal matrix, then all its eigenvalues are 1 or -1.

If $Uv = \lambda v$, then $\|v\| = \|Uv\| = \|\lambda v\| = |\lambda| \|v\|$.

So $|\lambda| = 1$, and $\lambda = \pm 1$.

4 pts

9. Define an operation on $\mathbb{R}^3$ by $\langle u, v \rangle := (Au) \cdot (Av)$. where $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.
Is this a valid inner product?

No — axiom 4 doesn't hold:

Suppose $\langle u, u \rangle = 0$. Then $(Au) \cdot (Au) = 0$, which means $Au = 0$, so $u \in \text{Nul } A$. But A has nontrivial nullspace, so u may not be 0.

e.g. Let $u = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. $\langle u, u \rangle = (Au) \cdot (Au) = 0 \cdot 0 = 0$.

5 pts

BONUS!

Let T be the linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$ that rotates space about the z-axis by 180°. Visualize this in your head.

(a) Find the eigenvalues of the matrix representation for T.
(Hint: you don't need to actually find the matrix. Think about this geometrically!)

$\lambda = 1$ (z-axis stays fixed) and $\lambda = -1$ (x-y plane gets negated).

2 pts

(b) Briefly explain why this matrix must be symmetric!

The e-spaces (z-axis and xy-plane) have dimensions adding to 3 and are orthogonal to each other, so we are ortho-diagonalizable $\therefore$ symmetric!

2 pts