MATH 54

MIDTERM I

NAME: $\boxed{\text{Solutions}}$

1	
2	
3	
4	
5	
6	
Total	

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1. Solution sets

(a) Find all solutions to the system

$$\begin{aligned} w + 3y &= 2 \\ x + 3z &= 3 \\ -2x + 3y + 2z &= 1 \\ w + 3x - 5z &= 4 \end{aligned}$$

(note, this is supposed to be a minus sign)

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 1 & 3 & 0 & -5 & 4 \end{array}\right] \xrightarrow[R3 := R3 + 2R2]{R4 := R4 - R1} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 3 & -3 & -5 & 2 \end{array}\right]$$

$$\xrightarrow{R4 := R4 - 3R2} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -3 & 4 & -7 \end{array}\right] \xrightarrow[R1 := R1 - R3]{R4 := R4 + R3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & -5 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & -5 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & -4/3 & 7/3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$w = -5 - 4z$
$x = 3 + 3z$
$y = 7/3 + 4/3\, z$
$z = \text{free}$

$$\text{Solution set} = \left\{ \begin{bmatrix} -5 \\ 3 \\ 7/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 3 \\ 4/3 \\ 1 \end{bmatrix} \right\}.$$

(b) Find the determinant of the coefficient matrix of the above system.

The determinant must be zero because the coefficient matrix reduces to something with a zero on the diagonal. (i.e, there are free variables, i.e. there aren't enough pivots.)

2 Linear Independence

Determine whether each set is linearly independent or dependent. Justify briefly.

(a) $\{\sin^2 x, -\cos^2 x, \pi\}$ in $C[0,1]$, the vector space of continuous functions on the interval $[0,1]$.

$\boxed{\text{Dependent}}$ $\sin^2 x + (-1)(-\cos^2 x) + (-\frac{1}{\pi})\pi$

$= \sin^2 x + \cos^2 x - 1 = 0$ by a trig identity.

(b) $\{1+x+2x^2,\ 3+x^2-x^3,\ 1-x^3,\ x-x^2,\ 2\}$ in P_3, the vector space of polynomials of degree at most 3.

$\boxed{\text{Dependent}}$: P_3 is a vector space of dimension 4, but we have 5 vectors. The largest possible linearly independent set in P_3 (i.e. a basis) has only 4 vectors in it.

(c) $\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\0&1\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}\right\}$ in $M_{2\times2}$ the space of 2x2 matrices.

$\boxed{\text{Independent}}$: Suppose $a\begin{bmatrix}1&0\\0&1\end{bmatrix} + b\begin{bmatrix}0&1\\1&0\end{bmatrix} + c\begin{bmatrix}0&1\\0&1\end{bmatrix} + d\begin{bmatrix}0&0\\1&1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$

i.e. $\begin{bmatrix}a & b+c\\ b+d & a+d\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$. i.e. $a=0$, $b+c=0$, $b+d=0$, $a+d=0$

i.e. $\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&1&0&0\\0&1&0&1&0\\1&0&0&1&0\end{array}\right] \xrightarrow[R3 := R3 - R2]{R4 = R4 - R1} \left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&-1&1&0\\0&0&0&1&0\end{array}\right]$, Pivots in every row means $a=b=c=d=0$ is the only solution!

(d) $\{[1]\}$ in the vector space V of positive real numbers with the operations of addition and scalar multiplication defined by
$[u]+[v] = [uv]$ and $c[u] = [u^c]$.

$\boxed{\text{Dependent}}$: As we learned in class, in this strange vector space, the zero vector $\vec{0}$ is actually $[1]$. Observe: $[u]+[1] = [u \cdot 1] = [u]$.

The zero vector is always dependent.

$\boxed{3}$ Inverses

(a) Find A^{-1}, where $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$, or if A^{-1} doesn't exist, explain why.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3 := R3 - 2R1]{R2 := R2 + 3R1} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R3 := R3 + 3R2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \xrightarrow[R2 := R2 + R3]{R1 := R1 + R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$$

Check: $$AA^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 8-7 & 3-3 & 1-1 \\ -24+10+14 & -9+4+6 & -3+1+2 \\ 16-30+14 & 6-12+6 & 2-3+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) True or False: $(A^TA)^{-1}(A^{-1})^T = A$ for any invertible matrix A.

$$(A^TA)^{-1}(A^{-1})^T = A^{-1}(A^T)^{-1}(A^{-1})^T = A^{-1}(A^T)^{-1}(A^T)^{-1} = A^{-1}(A^T \cdot A^T)^{-1}$$

$= (A^T \cdot A^T \cdot A)^{-1}$. However you simplify it, it just doesn't equal A. $\boxed{\text{False}}$

4 Linear Transforms

Let $T: P_2 \longrightarrow P_3$ be the transformation that maps $f(x) \longmapsto x\,f(x)$.

(a) Is T a linear transformation? If so, find the matrix representation for T.

Closed under $+$:

$$T(f(x)+g(x)) = x(f(x)+g(x)) = xf(x) + x\,g(x) = T(f(x)) + T(g(x))$$

closed under $\cdot$:

$$T(cf(x)) = x(cf(x)) = c(x\,f(x)) = c\,T(f(x)). \qquad \boxed{\text{Yes}}$$

$T(1) = x \qquad T(x) = x^2 \qquad T(x^2) = x^3$

so $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is our matrix for T.

(b) Is T one-to-one? Briefly explain.

$\boxed{\text{Yes}}$: A has a pivot in every column, i.e. no free variables.

(c) Is T onto? Briefly explain.

$\boxed{\text{No}}$: $\text{Col}\,A$ has dimension 3, but the codomain P_3 has dimension 4. So range $\neq$ codomain

Or, for example any constant function in P_3 has no preimage in P_2, so T isn't onto.

$\boxed{5}$ Subspaces

(a) Let H be the subset of vectors in $\mathbb{R}^5$ that read the same forward and backward (these are called <u>palindromic</u> vectors). For example, $\begin{bmatrix}1\\2\\3\\2\\1\end{bmatrix} \in H$. Is H a subspace of $\mathbb{R}^5$? If so, find a basis. If not, explain.

Contains $\vec{0}$: $\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$ is palindromic. ✓

closed under $+$: $\begin{bmatrix}a\\b\\c\\b\\a\end{bmatrix} + \begin{bmatrix}w\\x\\y\\x\\w\end{bmatrix} = \begin{bmatrix}a+w\\b+x\\c+y\\b+x\\a+w\end{bmatrix}$ is palindromic. ✓

closed under $\cdot$: $c\begin{bmatrix}w\\x\\y\\x\\w\end{bmatrix} = \begin{bmatrix}cw\\cx\\cy\\cx\\cw\end{bmatrix}$ is palindromic ✓

So, $\boxed{\text{Yes}}$.

To describe a vector in H we need to specify only the first 3 coordinates, so $\dim H = 3$ and a basis is

$$\left\{ \begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} \right\}$$

(b) Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

$\boxed{\text{No}}$: Vectors in $\mathbb{R}^2$ each have 2 coordinates, but vectors in $\mathbb{R}^3$ have 3. So $\mathbb{R}^2$ isn't even a <u>subset</u> of $\mathbb{R}^3$.

(c) Is P_2 a subspace of P_3?

$\boxed{\text{Yes}}$: Polynomials of degree ≤ 2 are also of degree ≤ 3, and since P_2 is a known vector space we know it's closed under $+$ and $\cdot$ and contains $\vec{0}$.

6 Special topic: Nilpotent matrices

A matrix B is nilpotent if $B^m = 0$ for some m.

(a) Can an invertible matrix be nilpotent? If so, give an example. If not, explain why.

$\boxed{\text{No}}$ Suppose $B^m = 0$. Look at the determinant of both sides:

$$(\det(B))^m = \det(B^m) = \det(0) = 0$$

So $\det(B)$ must be 0, which means B isn't invertible.

(b) Let $D: P_n \to P_{n-1}$ be the linear transformation of taking the derivative. Explain why the matrix representation for D is nilpotent.

Let A be the matrix of D.
Raising A to a power amounts to taking the derivative over and over.
i.e. A^n represents the linear transformation of taking the n^{th} derivative. Since a polynomial will eventually become 0 if we take enough derivatives, we can conclude that $A^m = 0$ for some m.
(In particular, for $T: P_n \to P_{n-1}$, $A^{n+1} = 0$.)