MATH 54

FINAL EXAM

NAME: Solutions

1	5
2	6
3	6
4	4
5	6
6	6
7	6
8	5
B	4
Total	44 + 4 bonus

+ INSTRUCTOR: JEFFREY DOKER +

1. Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$. Find a basis for Col A and a basis for Nul A.

Row reduce:

$$A \longrightarrow \begin{array}{c} \\ R2-2R1 \\ R3-R1 \end{array} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} 3R1 \\ 2R2 \\ \end{array} \begin{bmatrix} 3 & 6 & 3 & 0 \\ 0 & -6 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{array}{c} R1+R2 \\ \\ \end{array} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & -6 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \frac{1}{3}R1 \\ -\frac{1}{6}R2 \\ \end{array} \begin{bmatrix} 1 & 0 & 1/3 & 2/3 \\ 0 & 1 & 1/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivots in rows 1 and 2, so $\boxed{\text{Col } A \text{ basis} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}}$

$$\text{Nul } A = \left\{ \begin{pmatrix} -\frac{1}{3}s + -\frac{2}{3}t \\ -\frac{1}{3}s + \frac{1}{3}t \\ s \\ t \end{pmatrix} \right\} = \left\{ s \begin{pmatrix} -1/3 \\ -1/3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2/3 \\ 1/3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

so $\boxed{\text{basis} = \left\{ \begin{pmatrix} -1/3 \\ -1/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2/3 \\ 1/3 \\ 0 \\ 1 \end{pmatrix} \right\}}$.

5pts

2 Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Verify that 2 is an eigenvalue of A and v is an eigenvector.

Then orthogonally diagonalize A.

$$Av = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5v. \quad ✓$$

$$A - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ which is not invertible. ✓

$$E_2 = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} = \operatorname{span}\{u_1, u_2\}.$$

Orthogonal basis: $v_1 = u_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

$$v_2 = u_2 - \operatorname{proj}_{v_1} u_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}.$$

Orthonormal basis for $\mathbb{R}^3$:
$$\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$$

So
$$\boxed{A = SDS^T = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}}.$$

6 pts

3 Let $y'' + 12y' + 36y = e^{-6t}(\cos t - 1)$.

(a) Find a fundamental solution set to the associated homogeneous equation. Verify using the Wronskian.

Aux. equation: $r^2 + 12r + 36 = (r+6)(r+6)$. $r = -6, -6$.

solution: $\boxed{y_h = c_1 e^{-6t} + c_2 t e^{-6t}}$

$$W(t) = \begin{vmatrix} e^{-6t} & te^{-6t} \\ -6e^{-6t} & e^{-6t} - 6te^{-6t} \end{vmatrix}.$$

Let $t=0$, then $W(0) = \begin{vmatrix} 1 & 0 \\ -6 & 1 \end{vmatrix} = 1.$

so this is a f.s.s.

3 pts

(b) Find the form of a particular solution using undetermined coefficients.

$y'' + 12y' + 36y = e^{-6t}\cos t - e^{-6t}$.

The right side corresponds to roots $r = -6 \pm i$ and $r = -6$.

So our y_p will be of the form

$$\boxed{y_p = Ae^{-6t}\cos t + Be^{-6t}\sin t + t^2 Ce^{-6t}}$$

3 pts

4 Let λ be a real eigenvalue of A with eigenvector v. Prove that $x(t) = e^{\lambda t} v$ is a solution to $x' = Ax$.

Left side:

$$x'(t) = \lambda e^{\lambda t} v.$$

Right side:

$$Ax(t) = Ae^{\lambda t} v = e^{\lambda t}(Av) = e^{\lambda t}(\lambda v) = \lambda e^{\lambda t} v.$$

4 pts

5 A linear transformation T is represented by an $m \times n$ matrix A.

(a) How many pivots must A have if T is <u>onto</u>? Explain.

If T is onto, then the columns of A span $\mathbb{R}^m$. This means $\mathrm{Col}\, A = \mathbb{R}^m$, i.e. rank of $A = m$, so A has $\boxed{m \text{ pivots}}$ (a pivot in every <u>row</u>).

3 pts

(b) How many pivots must A have if T is <u>one-to-one</u>? Explain.

T is one-to-one iff $\mathrm{Nul}\, A = \{0\}$, i.e. there are no free variables. This means there is a pivot in every <u>column</u>, so A has $\boxed{n \text{ pivots}}$.

3 pts

6 Let $y'' - 2y' - 3y = 0$.

(a) Find the roots of the auxilary equation.

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r = -1, 3$

1 pt

(b) Express this differential equation as a first-order system $x' = Ax$, and solve it.

Let $x_1 = y$ $\quad x_1' = y' = x_2$

$x_2 = y'$ $\quad x_2' = y'' = 2y' + 3y = 2x_2 + 3x_1$.

$x'(t) = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} x(t)$, char polynomial of A = aux equation above, so $\lambda = -1, 3$.

$\underline{\lambda = -1}$: $A + I$: $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$. $E_{-1} = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$.

$\underline{\lambda = 3}$: $A - 3I$: $\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$. $E_3 = \text{span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}$

so $\boxed{x(t) = c_1 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}}$

3pts

(c) Explain briefly how to get a general solution of the original differential equation from the general solution of $x' = Ax$.

Since $x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$, we can simply take the first coordinate of $x(t)$ for our original solution: $y = c_1 e^{-t} \cdot (-1) + c_2 e^{3t} \cdot (1)$,

or $y = c_1 e^{-t} + c_2 e^{3t}$.

2 pts

$\boxed{7}$ (a) Compute e^{At} where $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

Diagonalize:

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 9 - 6\lambda + \lambda^2 - 1 = \lambda^2 - 6y + 8 = (\lambda - 2)(\lambda - 4)$$

$\lambda = 2, 4.$

$\underline{\lambda = 2}$: $A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $E_2 = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

$\underline{\lambda = 4}$: $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. $E_4 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

So $A = SDS^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

$$e^{At} = S e^{Dt} S^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -e^{2t} & e^{4t} \\ e^{2t} & e^{4t} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t}/2 + e^{4t}/2 & -e^{2t}/2 + e^{4t}/2 \\ -e^{2t}/2 + e^{4t}/2 & e^{2t}/2 + e^{4t}/4 \end{bmatrix}.$$

4 pts

(b) Find a vector $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ such that $e^{At}c$ is the solution to the IVP $x'(t) = Ax(t)$, $x(0) = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$.

We want to find c such that $x(0) = e^{A \cdot 0} \cdot c = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$.

But we can see that $e^{A \cdot 0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $x(0) = I \cdot c = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$,

and we conclude $c = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$.

2 pts

$\boxed{8}$ (a) The four s-words for graphing trajectories are

sources, sinks, saddles, and spirals. 1 pt.

(b) Let $x' = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} x$. Determine if solutions to this system spiral in or out, and determine if they spiral clockwise or counterclockwise.

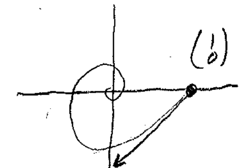
Char poly: $|A - \lambda I| = \begin{vmatrix} -1-\lambda & 2 \\ -1 & -3-\lambda \end{vmatrix} = (1+\lambda)(3+\lambda) + 2 = \lambda^2 + 4\lambda + 5.$

$\lambda = \frac{-4 \pm 2i}{2} = -2 \pm i$. Since the real part of λ, -2, is negative,

we have a $\boxed{\text{spiral in}}$.

Test for cw/ccw: Take $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $x'(0) = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

So the slope of our guess point is pointed like this:



And we conclude the spiral is $\boxed{\text{clockwise.}}$

4 pts

$\boxed{\text{Bonus}}$ Let $f(x)$ be a triangle wave defined by $f(x) = |x|$ on $[-1, 1]$.

(a) Calculate a_0 of the Fourier series of $f(x)$.

$$a_0 = \frac{1}{T}\int_{-T}^{T} |x|\, dx = \int_{-1}^{1} |x|\, dx = \underbrace{2 \cdot \int_0^1 |x|\, dx}_{\text{(even function)}} = \underbrace{2\int_0^1 x\, dx}_{(|x| = x \text{ when } x \geq 0)} = 2 \cdot \frac{x^2}{2}\Big|_0^1 = \boxed{1}$$

2 pts

(b) Other than a_0, the Fourier series of $f(x)$ contains:

only sines, $\boxed{\text{only cosines}}$ or some of each.

(Circle one and explain.)

Since $f(x) = |x|$ is an even function, $|x| \sin\frac{n\pi x}{T}$ is odd, so $b_n = 0$ for all n.

On the other hand, $|x| \cos\frac{n\pi x}{T}$ is even, so $a_n = \frac{2}{T}\int_0^T |x| \cos\frac{n\pi x}{T}$, which isn't zero.

2 pts.

I had such a wonderful time getting to know you all this summer. I hope you had fun too. Good luck in school and in life, and always remember that you are brilliant! – J.D.