MATH 74 HOMEWORK 5 SOLUTIONS

1. Just so people can see how these go I am including proofs. They were not a required part of the homework.

1(a). Injective but not surjective.

Proof. To see that f is not surjective, notice that 2 is not in the range of f. Indeed, $1^2 = 1$ is less than 2, and if $x \geq 2$, then $x^2 \geq 4$ is bigger than 2, so there is no $n \in \mathbb{N}$ for which $n^2 = 2$.

To see that f is injective, fix $x, y \in \mathbb{N}$ and suppose $f(x) = f(y)$, ie, that $x^2 = y^2$. We cannot have $x < y$ because then $x^2 = x \cdot x < x \cdot y < y \cdot y = y^2$, and similarly (interchanging x and y in the proof just given) we cannot have $x > y$. So $x = y$. □

There is a general fact at work here. The point is that squaring, on the given domain, is *strictly increasing*: if $x < y$, then $f(x) < f(y)$. Any strictly increasing function is injective.

1(b). Injective but not surjective.

Proof. To see that it is injective, consider the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^{1/3}$. Then

$$g(f(x)) = (x^3)^{1/3} = x, \qquad x \in \mathbb{R}.$$

so f is injective by "Theorem 18" from class.

To see that f is not surjective, notice that every number in the range of f, being a cube of a nonnegative number, must be nonnegative. So any negative number (eg -1) is not in the range of f. □

1(c). Injective and surjective.

Proof. Without a precise definition of the sine function (or the number π) we are not really in a position to "prove" this, but it is clear from calculus. On the given domain, $f(x)$ is an increasing function of x, so f is injective, and $f(-\pi/2) = -1$ and $f(\pi/2) = 1$, so (by continuity of sin and the intermediate value theorem) it is surjective also. □

1(d). Neither injective nor surjective.

Proof. To see that f is not injective, notice that $f(\{1\}) = \{1, 2, 3\} = f(\{2\})$, but of course $\{1\} \neq \{2\}$.

To see that f is not surjective, notice that by definition of f we have $1 \in f(A)$ for any $A \in \mathcal{P}(\mathbb{N})$. So for example $A = \{2\}$ (or any other set without 1 in it) is not in the range of f. □

1(e). Injective and surjective.

Proof. A quick way to prove this is to convince yourself that

$$f(f(A)) = A$$

for all $A \in \mathcal{P}(\mathbb{N})$. Thus f is a bijection by "Theorem 20" from class (with the role of g played by f). □

2. In this exercise we define a function $\phi : \mathcal{P}(A) \to B$ and prove it is a bijection. Since the elements of B are themselves functions, the usual notation for the application of ϕ, ie $A \mapsto \phi(A)$, looks confusing— since $\phi(A)$ is itself a function, and can be evaluated at things. So we will denote the application of ϕ by $X \mapsto \phi_X$ instead.

Proof. Consider the function $\phi : \mathcal{P}(A) \to B$ given by the following rule. Given a subset X of A, let define the function $\phi_X \in B$ by

$$\phi_X(a) = \begin{cases} 0 & a \notin X \\ 1 & a \in X \end{cases}$$

We will now show that ϕ is injective. If X and Y are subsets of A and $\phi_X = \phi_Y$, we have

$$\begin{aligned} X &= \{a \in A : \phi_X(a) = 1\} && \text{by definition of } \phi_X \\ &= \{a \in A : \phi_Y(a) = 1\} && \text{since } \phi_X = \phi_Y \\ &= Y && \text{by definition of } \phi_Y \end{aligned}$$

Since X and Y were arbitrary we conclude ϕ is injective.

We now show that ϕ is surjective. If $g : A \to \{0, 1\}$ is any element of B, let $X = \{a \in A : g(a) = 1\}$. By definition we have that $\phi_X(a) = 1$ when $g(a) = 1$, and that $\phi_X(a) = 0$ when $g(a) = 0$. In other words, $g(a) = \phi_X(a)$ for all $a \in A$. This means that $g = \phi_X$, so that g is in the range of ϕ. Since g was arbitrary, ϕ is surjective. □

3. 3(a) Yes, f is injective. (Again, without a precise definition of e^x it is hard to prove, but take some calculus for granted. It is an increasing function. No horizontal line intersects its graph twice.)

3(b) $f|_S$ is injective for any subset S of $\mathbb{R}$. (If $f|_S$ failed to be injective, then f would fail to be injective, too.)

3(c) No, f is not surjective. ($-1 \in \mathbb{R}$ is not in the range of f.)

3(d) There are no subsets S of $\mathbb{R}$ for which f_S is surjective. (If f_S were surjective, then f would be too)

4(a). There are $\#(B)$ functions from a one-element set into B.

4(b). We need to show that ϕ is injective. Suppose that f and g are two functions $A \to B$ such that $\phi(f) = \phi(g)$. This means that

$$(f|_{A\setminus\{a\}}, f(a)) = (g|_{A\setminus\{a\}}, g(a)).$$

This means that $f|_{A\setminus\{a\}} = g|_{A\setminus\{a\}}$ and $f(a) = g(a)$. In other words, $f(x) = g(x)$ for all $x \in A \setminus \{a\}$, and also $f(a) = g(a)$. We conclude that $f(x) = g(x)$ for all $x \in A$, that is, $f = g$. As f and g were arbitrary, we conclude that ϕ is injective.

To show that ϕ is surjective, fix an element of $T \times B$. This is a pair (g, b) with $g : A \setminus \{a\} \to B$ and $b \in B$. Defining $f : A \to B$ by

$$f(x) = \begin{cases} g(x) & x \neq a \\ b & x = a \end{cases}$$

We see that $f|_{A \setminus \{a\}} = g$ and $f(a) = b$; in other words, $\phi(f) = (g, b)$. Since (g, b) was arbitrary, we conclude ϕ is surjective.

4(c). Notice that this means $\#(S) = \#(T) \cdot \#(B)$.

[Indeed, in 4(a) we exhibited a bijection between S and $T \times B$, so the two sets have the same number of elements. Since $\#(T \times B) = \#(T) \times \#(B)$, that formula follows.]

Using this formula and 4(a), we see that if A has two elements, the number of functions $A \to B$ is $\#(B) \cdot \#(B) = (\#(B))^2$, and so on; the number of functions from $A \to B$ ought to be $(\#(B))^{\#(A)}$.

We will now prove this.

Theorem. *If A and B are finite sets, then the number of functions from $A \to B$ is $(\#(B))^{\#(A)}$.*

To prove this we use the induction principle. The statement $P(n)$ we use is "if a set has n elements, the number of functions from it into B is $\#(B)^n$."

Proof. The statement $P(1)$ is true from 4(a).

Assume that $k \in \mathbb{N}$ is such that $P(k)$ is true, and let A be any set of $k + 1$ elements. Picking an element $a \in A$ and applying 4(b),

$$\#\{\text{functions } A \to B\} = \#\{\text{functions } A \setminus \{a\} \to B\} \times \#(B),$$

but since $A \setminus \{a\}$ has k elements, our hypothesis $P(k)$ tells us that

$$\#\{\text{functions } A \setminus \{a\} \to B\} = \#(B)^k$$

and we conclude

$$\#\{\text{functions } A \to B\} = \#(B)^k \cdot \#(B) = \#(B)^{k+1}$$

proving $P(k + 1)$. By the induction principle the statement is true for all n. $\square$

5. We have

$$\begin{aligned} \#(\mathcal{P}(A)) &= \#\{\text{functions } A \to \{0, 1\}\} && \text{by Exercise 2} \\ &= 2^{\#(A)} && \text{by 4(c)} \end{aligned}$$