Here are sketches of the solutions.

There are multiple ways to answer most of the problems, I present just one way.

Since I am sketching and not taking the exam, some of my proofs skip over details I would rather you had written out.

All theorem numbers refer to the numbering on the theorem sheet on my webpage (slightly different from the numbering at the actual final).

FREE RESPONSE

1. $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5, 6\}$, $C = \{4, 5, 6\}$ will do.

2. $a = c = 2$, $b = 1$ will do.

3. $f(n) = \begin{cases} 2n+1 & n \geq 0 \\ 2(-n) & n < 0 \end{cases}$ will do (in fact it is a bijection $\mathbb{Z} \to \mathbb{N}$).

4. $a = b = c = 2$ will do.

5. There are no examples. (If a_n converges to a, then a_{n+1} converges to a also by Theorem 9, and then $a_{n+1} - a_n$ is convergent to $a - a = 0$ by Theorem 8.)

6. $c_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ will do. (It was shown in class on May 2 that $c_n \geq \sqrt{n}$, so c_n cannot converge.) Alternatively $c_n = \log n$, or $c_n = \sqrt{n}$, or ...)

TRUE/FALSE

1. True. (This was stated in class on April 30. On two homeworks we proved that at least the value of $L(30)$ is determined by the given properties. The same goes for any other value of L.)

2. False. (Example: 1 is not in the range of f. More generally you can show that $f(n)$ is always even, so no odd number is in the range of f.)

3. False. (Example: $a = 4$, $b = 2$, $c = 4$.)

4. True. (Shown in problem 2 of the third homework, and discussed in class on February 12.)

5. True. (This appeared on the preview of midterm 2.)

6. True. (Proved in class on May 2.)

7. False. (Example: $(-1)^n$ is bounded above by 1 and below by -1, but not convergent; done in lecture on April 2.)

8. True. (Stated several times in class, e.g. May 4. Can be proved straight from the definition, or is "clear" from intuition about convergence.)

9. True. (The given inequality implies that $s_n \leq 51$ for all n, so that s_n, an increasing sequence, is bounded above, hence convergent by Theorem 5.)

PROOFS

1. It is clear enough that $\{1, p, q, pq\} \subseteq D(pq)$ so we focus on showing that $D(pq) \subseteq \{1, p, q, pq\}$.

Let d be any divisor of pq. Consider $\gcd(d, p)$. Since it is a divisor of p, and p is prime, it must either be 1 or p.

If $\gcd(d, p) = 1$, then by Theorem 1 on the sheet we conclude $d \mid q$, which means $d = 1$ or $d = q$ since q is prime.

If $\gcd(d, p) = p$, then $p \mid d$, so there is $k \in \mathbb{N}$ with $d = pk$. Since $d \mid pq$ there is $l \in \mathbb{N}$ with $pq = dl$. We conclude that $pq = pkl$ and hence that $q = kl$. This means $k \mid q$, so $k = 1$ or q since q is prime. Then $d = pk$ is either p or pq.

Conclusion: no matter what, d is one of the numbers $1, q, p, pq$.

[This question was on the final preview. It was discussed but not solved in lecture on January 29 and 31.]

2(a). We do this from the definition. Suppose (x, y) and (u, v) in $\mathbb{Z} \times \mathbb{Z}$ satisfy $g(x, y) = g(u, v)$. This means (i) $x - y = u - v$ and (ii) $x + y = u + v$. Adding (i) and (ii) we see that $2x = 2u$ so that $x = u$; then a look at either (i) or (ii) shows us that $y = v$. Conclusion: if $g(x, y) = g(u, v)$, then $(x, y) = (u, v)$. This shows g is injective.

[A minor variation on this was on the final preview.]

2(b). We prove that $(0, 1)$ is not in the range of g. If there are $x, y \in \mathbb{Z}$ with $g(x, y) = (0, 1)$, then (i) $x - y = 0$ and (ii) $x + y = 1$. The equation (i) implies that $x = y$, and then (ii) tells us $2x = 1$. But no $x \in \mathbb{Z}$ can satisfy this equation, so g is not surjective.

[The same idea shows that no (x, y) with $x + y$ odd is in the range of f. You would also be led to this if you tried calculating a formula for the inverse of g; the formula requires division by 2, which is problematic for many integers.]

[A minor variation on this was on the final preview.]

3. Let A be the set of subsets with an even number of elements and B the set of subsets with an odd number of elements, and define $f : A \to B$ by $f(X) = \begin{cases} X \setminus \{1\} & 1 \in X \\ X \cup \{1\} & 1 \notin X \end{cases}$. We can use the same rule to define a function $g : B \to A$ which can be seen to satisfy both $g(f(X)) = X$ for all $X \in A$ and $f(g(X)) = X$ for all $X \in B$. Theorem 3 on the cheat sheet implies that f is a bijection and that $\#A = \#B$ as desired.

[Many other solutions possible— among them: just writing out all subsets and counting. The complementation map $X \mapsto \{1, 2, 3, 4, 5\} \setminus X$ is also a bijection $A \to B$.]

[This problem is a special case ($n = 5$) of problem I on midterm 2, and 1(b) on homework 6.]

4. Define a sequence a_n for all $n \in \mathbb{N}$ by

$$a_n = \sum_{k=1}^{n} (2k - 1).$$

4(a). Let $b_n = n^2$. Notice that $a_1 = 1 = b_1$ and that for any $n \in \mathbb{N}$ we have

$$b_{n+1} - b_n = (n+1)^2 - n^2 = 2n + 1 = 2(n+1) - 1 = a_{n+1} - a_n.$$

From Theorem 4 we conclude that $a_n = b_n$ for all n, that is, $a_n = n^2$ for all $n \in \mathbb{N}$.

4(b). By the distributive law,

$$a_n = \sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n}(2k) - \sum_{k=1}^{n} 1 = 2\sum_{k=1}^{n} k - n = 2c_n - n$$

Solving for c_n we conclude that $c_n = \frac{a_n+n}{2} = \frac{n^2+n}{2}$ for all $n \in \mathbb{N}$.

5(a). See the lecture notes from April 2.

5(b). Algebra shows us that for every $n \in \mathbb{N}$ we have

$$x_n = (\sqrt{n^2+n} - n)\left(\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}\right) = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+1/n}+1}$$

We will use several times that the constant sequence 1 converges (by Theorem 7).

This and 5(a) together with Theorem 8 tells us that $1+1/n$ converges to 1. The fact given in the problem implies that $\sqrt{1+1/n}$ converges to 1. Then Theorem 8 tells us that $\sqrt{1+1/n}+1$ converges to $1+1=2$.

Since the numerator sequence 1 is convergent (Theorem 7), and the denominator sequence (which is a sequence of nonzero numbers) is convergent to 2 (which is nonzero), Theorem 8 tells us that x_n converges to $\frac{1}{2}$.

[This was a lot like problem 8 on homework 7, and similar problems were discussed in lecture on March 23.]

6. See the lecture notes from April 4.