

Math 222-4 Modterm Exam I: Version B

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Discussion Session TA:

If you find any errors, please let your TA know 

1. (32 points) Evaluate

$$(a) \int x^{-1}(\ln x)^{n+1} dx \quad n \geq -1; \quad (b) \int \frac{e^{-x} dx}{e^{-2x} + 2e^{-x} + 2}; \quad (c) \int \frac{1}{\sqrt{2x-x^2}} dx$$

$$(d) \int e^x \sin(x) + \ln(x) dx; \quad (\text{Hint: treat two terms separately})$$

$$(a) u = \ln x$$

$$du = \frac{1}{x} dx$$

$$\int u^{n+1} du = \frac{u^{n+2}}{n+2} + C$$

$$= \frac{(\ln x)^{n+2}}{n+2} + C$$

$$(c) 2x - x^2 = -(x^2 - 2x) \\ = -(x-1)^2 + 1$$

$$? = \int \frac{1}{\sqrt{1-(x-1)^2}} dx$$

$$= \arcsin(x-1) + C$$

$$(b) u = e^{-x} \quad du = -e^{-x} dx \\ -du = e^{-x} dx$$

$$? = \int \frac{-du}{u^2 + u + 2} = - \int \frac{du}{(u+1)^2 + 1}$$

$$= -\arctan(u+1) + C$$

$$= -\arctan(e^{-x} + 1) + C$$

$$(d) I = \int e^x \sin x dx \quad F = \sin x \quad G' = e^x \\ F' = \cos x \quad G = e^x$$

$$= e^x \sin x - \int e^x \cos x dx \quad F = \cos x \quad G' = e^x \\ F' = -\sin x \quad G = e^x$$

$$= e^x \sin x - (\cos x \cdot e^x - \int \sin x \cdot e^x dx)$$

$$= e^x \sin x - e^x \cos x - \underbrace{\int \sin x e^x dx}_I$$

$$\Rightarrow 2I = e^x \sin x - e^x \cos x$$

$$\Rightarrow I = \frac{e^x \sin x - e^x \cos x}{2} + C$$

$$\int \ln x dx = x \ln x - \int \frac{1}{x} \cdot x dx \quad F = \ln x \quad G' = 1 \\ F' = \frac{1}{x} \quad G = x \\ = x \ln x - x + C$$

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$$\Rightarrow ? = \frac{e^x \sin x - e^x \cos x}{2} + x \ln x - x + C$$

2. (20 points) Evaluate improper integrals:

$$(a) \int_1^{\infty} \frac{1}{x^{3/2}} + \frac{1}{x} dx; \quad (b) \int_{-\infty}^0 e^{\sqrt{-x}} dx$$

$$\begin{aligned} (a) & \int \frac{1}{x^{3/2}} + \frac{1}{x} dx \\ &= \int x^{-\frac{3}{2}} + \frac{1}{x} dx \\ &= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + \ln|x| + C = -2x^{-\frac{1}{2}} + \ln|x| + C \end{aligned}$$

$$\begin{aligned} ? &= \lim_{a \rightarrow \infty} \int_1^a \frac{1}{x^{3/2}} + \frac{1}{x} dx \\ &= \lim_{a \rightarrow \infty} \left( -2x^{-\frac{1}{2}} + \ln|x| \right) \Big|_{x=1}^{x=a} \\ &= \lim_{a \rightarrow \infty} \frac{-2}{\sqrt{a}} + \ln|a| - (-2+0) \end{aligned}$$

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(Because  $\lim_{a \rightarrow \infty} |\ln|a|| = +\infty$ )

$$\begin{aligned} (b) & u = -\sqrt{-x} \\ & u^2 = -x \\ & 2u du = -dx \Rightarrow -2u du = dx \\ & \int e^{-\sqrt{-x}} dx = -2 \int u e^u du \quad F=u \quad G'=e^u \\ & \quad F'=1 \quad G=e^u \\ &= -2 \left[ ue^u - \int e^u du \right] \\ &= -2 \left[ ue^u - e^u \right] + C \\ &= +2 \sqrt{-x} \cdot e^{-\sqrt{-x}} + 2e^{-\sqrt{-x}} + C \end{aligned}$$

$$\begin{aligned} ? &= \lim_{a \rightarrow -\infty} \int_a^0 e^{-\sqrt{-x}} dx \\ &= \lim_{a \rightarrow -\infty} \left( 2\sqrt{-x} \cdot e^{-\sqrt{-x}} + 2e^{-\sqrt{-x}} \right) \Big|_{x=a}^{x=0} \\ &= \lim_{a \rightarrow -\infty} 2 - \left( \frac{2\sqrt{-a}}{e^{\sqrt{-a}}} + \frac{2}{e^{\sqrt{-a}}} \right) \\ &= 2 - 0 = 2 \boxed{\text{exponential grows polynomials faster than}} \end{aligned}$$

3. (16 points) Solve the following differential equations:

$$\frac{dy}{dx} = -y \tan x + 2 \cos x, \quad y(0) = 0.$$

$$\frac{dy}{dx} + y \tan x = 2 \cos x$$

$$a(x) = \tan x \quad b(x) = 2 \cos x$$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx \quad \begin{matrix} u = \cos x \\ du = -\sin x dx \end{matrix} \quad - \int \frac{1}{u} du$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$m(x) = e^{-\ln|\cos x|} = \frac{1}{|\cos x|} \quad \text{choose } \frac{1}{\cos x}$$

$$\frac{d}{dx} \left[ \frac{1}{\cos x} y \right] = \frac{1}{\cos x} \cdot 2 \cos x$$

$$\frac{1}{\cos x} y = \int 2 dx = 2x + C$$

$$y = \cos x (2x + C)$$

Plug in  $(x, y) = (0, 0)$ ,  $0 = 0 + C \Rightarrow C = 0$

$$y = \cos x \cdot 2x$$

4. (32 points) Let  $P(t)$  be the size of population in an island. Assume that the size of the population changes according to the so-called logistic equation

$$\frac{dP}{dt} = 0.5P(500 - P)$$

Assume also that in the beginning the population size is  $P=100$ .

- (a) Find the general solution to the differential equation (assume you know  $0 < P < 500$ ).
- (b) Find the solution that satisfies the given initial conditions
- (c) How long does it take the population to reach size  $P = 250$
- (d) What value does  $P$  have when  $dP/dt$  is the largest (hint: you do not need to solve the differential equation—this question has a very short answer).

(a) Separable.

$$\frac{dp}{P(500-P)} = 0.5dt$$

$$\frac{A}{P} + \frac{B}{500-P} = \frac{1}{P(500-P)}$$

$$A(500-P) + BP = 1$$

$$\text{Plug in } P=0 \Rightarrow A = \frac{1}{500}$$

$$P=500 \Rightarrow B = \frac{1}{500}$$

$$\Rightarrow \int \frac{1}{500} + \frac{1}{500-P} dp = \int 0.5 dt$$

$$\frac{1}{500} \ln|P| - \frac{1}{500} \ln|500-P| = 0.5t + C$$

$$\frac{1}{500} \ln \left| \frac{P}{500-P} \right| = 0.5t + C$$

$$\ln \left| \frac{P}{500-P} \right| = 250t + 500C$$

$$\left| \frac{P}{500-P} \right| = e^{250t} \cdot e^{500C}$$

$$\frac{P}{500-P} = \underbrace{\pm e^{500C}}_{\text{if denote as }} \cdot e^{250t} \quad (*)$$

$$P = Ae^{250t}(500-P) = Ae^{250t} \cdot 500 - Ae^{250t}P$$

$$(1+Ae^{250t})P = Ae^{250t} \cdot 500$$

$$\Rightarrow P = \frac{Ae^{250t} \cdot 500}{1+Ae^{250t}} \quad \text{General solution.}$$

$$(b) P(0)=100 \quad t=0, P=100$$

Plug in to Eq (\*)

$$\frac{1}{4} = A e^0 \Rightarrow A = \frac{1}{4}$$

$$P = \frac{\frac{1}{4} \cdot e^{250t} \cdot 500}{1 + \frac{1}{4} e^{250t}}$$

$$(c) \text{ Eq (*) is } \frac{P}{500-P} = \frac{1}{4} e^{250t}$$

Plug in  $P=250$

$$\frac{250}{250} = \frac{1}{4} e^{250t}$$

$$4 = e^{250t}$$

$$\ln 4 = 250t$$

$$t = \frac{\ln 4}{250}$$

$$(d) \text{ Denote } \frac{dP}{dt} = 0.5P(500-P) \triangleq f.$$

Want  $f$  be largest, i.e.  $f' = 0$ .

$$f' \xrightarrow{\text{product rule}} 0.5(500-P) + 0.5P(-1)$$

$$= 0.5(500 - 2P) = 0$$

$$\Rightarrow P = 250.$$