REPRESENTATION THEORY OF SYMMETRIC GROUPS AND HECKE ALGEBRAS

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1. The Arakawa-Suzuki Functor

1.1. **Preliminary Notation.** Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ be the Lie algebra of $n \times n$ matrices with basis of matrix units $\{e_{ij} | 1 \leq i, j \leq n\}$. The triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is such that $\mathfrak{n}^+$ is the set of upper triangular matrices, $\mathfrak{h}$ the set of diagonal matrices, and $\mathfrak{n}^-$ the set of lower triangular matrices. Let $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ be the linear dual of $\mathfrak{h}$, and ε_i be the ith coordinate function: $\varepsilon_i(e_{jj} = \delta_{ij}$. Then we have the inclusions $\Delta \subset R^+ \subset R \subset Q \subset P \subset \mathfrak{h}^*$ where

- $\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} | 1 \leq i < n\}$ are the simple roots;
- $R^+ = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j | 1 \leq i < j \leq n\}$ are the positive roots relative to Δ;
- $R = R^+ \cup (-R^+)$ is the root system;
- $Q = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i$ is the root lattice; and
- $P = \bigoplus_{i=1}^{n} \mathbb{Z}\omega_i$ is the weight lattice, where $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ is the ith fundamental dominant weight.

Given a root $\alpha = \alpha_{ij} = \varepsilon_i - \varepsilon_j$, let $\mathfrak{h}_\alpha = e_{ii} - e_{jj}$, $h_i = h_{\alpha_i}$. Then, the fundamental dominant weights satisfy $\omega_i(h_j) = \delta_{ij}$. We have included the fundamental dominant weight ω_n which vanishes on all h_i. This is simply here to make formulae "nice" and will not be used outside of the next paragraph.

We can and will identify $P = \mathbb{Z}^n = \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i$ ($\varepsilon_i = \omega_i - \omega_{i-1}$, $1 < i \leq n$) as the set of n-tuples of integers. Let $P^+ = \bigoplus_i \mathbb{Z}_{\geq 0}\omega_i$ be the set of dominant integral weights.

Exercise 1.1.1. *Under the identification $P = \mathbb{Z}^n$, the set P^+ corresponds to the set of partitions of length at most n.*

We will also need to consider the element $\rho = \omega_1 + \cdots + \omega_{n-1} = \frac{1}{2}\sum_{\alpha \in R^+} \alpha = (n-1, \ldots, 1, 0)$. The symmetric group S_n acts on $P = \mathbb{Z}^n$ via permuting the coordinates: If $\lambda = (\lambda_1, \cdots, \lambda_n) \in P$ and a simple transposition $s_i = (i, i+1) \in S_n$, $s_i(\lambda) = (\lambda_1, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_n$. There is also the *dot action* of the symmetric group on P: given $\lambda \in P$ and $w \in S_n$, $w \circ \lambda = w(\lambda - \rho) + \rho$.

Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. The Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$ is the category of all $\mathfrak{g}$-modules M such that

1. M is finitely generated over $U(\mathfrak{g})$;
2. M is $\mathfrak{h}$-diagonalizable, i.e.
$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda \text{ where } M_\lambda = \{v \in M | h.v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\};$$
3. M is $\mathfrak{n}^+$-locally finite, i.e. $\dim U(\mathfrak{n}^+)v < \infty$ for all $v \in M$.

Morphism are $\mathfrak{g}$-module homomorphisms.

We call $\lambda \in \mathfrak{h}^*$ a *weight* of M if $M_\lambda \neq 0$, and define $P(M)$ to be the set of all weights of M. From now on, we will work with the full subcategory $\mathcal{O}_{\mathrm{int}}$ of $\mathcal{O}$ of modules M satisfying $P(M) \subset P$, but for convenience, we drop the subscript int.

As is well known, the Grothendieck Group $K(\mathcal{O})$ has a basis given by *Verma Modules*. Given $\lambda \in P$, the Verma module of highest weight λ is as follows: Let $\mathbb{C}v_\lambda$ be the 1-dimensional $\mathfrak{h}$-module with $h.v_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}$. This module may be regarded as a $\mathfrak{h} \oplus \mathfrak{n}^+$-module via $x.v_\lambda = 0$ for all $x \in \mathfrak{n}^+$. Then,

$$M(\lambda) = \mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}^+}^{\mathfrak{g}} \mathbb{C}v_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}^+)} \mathbb{C}v_\lambda \cong U(\mathfrak{n}^-).v_\lambda.$$

It is well known (and easy to prove) that this module is free as a $U(\mathfrak{n}^-)$-module, and $M(\lambda)_\lambda$ is 1-dimensional.

Now, consider the center $\mathcal{Z} \subset U(\mathfrak{g})$. A *central character* is a homomorphism $\chi : \mathcal{Z} \to \mathbb{C}$. Let $\mathcal{O}^{[\chi]}$ denote the *block* of the category $\mathcal{O}$ corresponding to the central character χ. That is, M is an object in $\mathcal{O}^{[\chi]}$ if and only if for all $z \in \mathcal{Z}$, $(z - \chi(z))^N M = 0$ for $N \gg 0$.

We now construct characters from Verma modules. Indeed, let $\lambda \in P$ and $M(\lambda)$ the corresponding Verma module. Given $z \in \mathcal{Z}$, $z.v_\lambda$ is another vector of weight λ since z commutes with the action of $\mathfrak{h} \oplus \mathfrak{n}^+$. Since $M(\lambda)_\lambda$ is 1-dimensional, it follows that $z.v_\lambda$ is a scalar multiple of v_λ. Denote this scalar $\chi_\lambda(z)$. In this way we have defined a central character χ_λ. It is known that

(1) $\{\chi_\lambda | \lambda \in P\}$ is a complete set of central characters;
(2) $\chi_\lambda = \chi_\mu$ if and only if $\lambda = w \circ \mu$ for some $w \in S_n$.

This means that blocks of $\mathcal{O}$ are labelled by S_n-orbits of P under the dot action. Write $\mathcal{O}^{[\lambda]} := \mathcal{O}^{[\chi_\lambda]}$. Then, given $M \in \mathcal{O}$,

$$M = \bigoplus_{\lambda \in P/(S_n,\circ)} M^{[\lambda]}, \quad M^{[\lambda]} \in \mathcal{O}^{[\lambda]}$$

since one may use elements of $\mathcal{Z}$ to define projections onto individual blocks.

1.2. **The Functor.** Our goal now is to define a homomorphism

$$\mathcal{H}^{\mathrm{aff}}(d) \to \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d}), \tag{1}$$

where $V = \mathbb{C}^n$ is the vector representation of $\mathfrak{g}$, M is (a priori) any $\mathfrak{g}$-module, and $\mathcal{H}^{\mathrm{aff}}(d)$ is the (degenerate) affine Hecke algebra of S_n. Throughout this discussion, we will think of M as inhabiting the 0th tensor place, while $V^{\otimes d}$ occupies the tensor places 1 through d.

The action of $\mathcal{H}^{\mathrm{aff}}(d)$ on this space will come from the Casimir element Ω of $\mathfrak{g}$, where we define

$$\Omega = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}.$$

To explain the role of the Casimir in this construction, consider the case $M = \mathbb{C}$ is the trivial representation, and $d = 2$. Then,

$$\Omega \in \mathrm{End}_{\mathbb{C}}(V \otimes V)$$

is the permutation operator $\Omega.(v \otimes v') = v' \otimes v$.

Exercise 1.2.1. *(1) The Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ acts on $V \otimes V$ by*

$$x.(v \otimes v') = (Xv) \otimes v' + v \otimes (Xv').$$

Show that $\Omega x = x\Omega$ as operators on $V \otimes V$, and so $\Omega \in \mathrm{End}_{\mathfrak{g}}(V \otimes V)$.

(2) Show that the subalgebra of $\mathrm{End}_{\mathfrak{g}}(V \otimes V)$ generated by Ω is isomorphic to the group algebra of the symmetric group S_2. Hence there is an injective homomorphism

$$\phi : S_2 \hookrightarrow \mathrm{End}_{\mathfrak{g}}(V \otimes V).$$

(3) Show that if $n \geq 2$, ϕ is surjective. Hence,

$$\mathbb{C}S_2 \cong \mathrm{End}_{\mathfrak{g}}(V \otimes V).$$

More generally, the symmetric group acts on $V^{\otimes d}$ by place permutation and the transposition $s_{ij} = (ij)$ is realized by the operator $S_{ij} = \Omega_{ij} \in \mathrm{End}_{\mathbb{C}}(V^{\otimes d})$ which just the Casimir acting diagonally in the ith and jth tensor positions:

$$S_{ij} = 1^{\otimes i-1} \otimes e_{ij} \otimes 1^{j-i-1} \otimes e_{ji} \otimes 1^{d-j}.$$

Theorem 1.2.2. *(Schur) If $n \geq d$, there exists an isomorphism*

$$\phi : \mathbb{C}S_d \to \mathrm{End}_{\mathfrak{g}}(V^{\otimes d}), s_{ij} \mapsto S_{ij}.$$

Now, to get the Hecke algebra into the picture, we introduce the 0th tensor place.

Exercise 1.2.3. *Let $X_1 = \Omega_{01} \in \mathrm{End}_{\mathbb{C}}(M \otimes V \otimes V)$, and define $X_2 = S_{12}X_1S_{12} + S_{12}$. Show that*

(1) *Show that $S_{12}X_1S_{12} = \Omega_{12}\Omega_{01}\Omega_{12} = \Omega_{02}$.*

(2) *$X_1X_2 = X_2X_1$. (Hint: this is equivalent to showing that*

$$\Omega_{01}\Omega_{02} - \Omega_{02}\Omega_{01} = (\Omega_{02} - \Omega_{01})\Omega_{12}$$

Which is an easy matrix mutiplication problem.)

(3) *The operators X_1 and X_2 commute with the action of $\mathfrak{g}$ and so there exists a homomorphism*

$$\phi : \mathcal{H}^{\mathrm{aff}}(2) \to \mathrm{End}_{\mathfrak{g}}(M \otimes V \otimes V).$$

(Hint: It is enough to show that X_1 commutes with the action of $\mathfrak{g}$. This amounts to showing that $[\Omega, 1 \otimes X + X \otimes 1] = 0$ for every $X \in \mathfrak{g}$. You might as well take X to be a matrix unit.)

More generally, for $1 \leq i \leq d$, define $X_i = \Omega_{0i} + \cdots + \Omega_{i-1,i} \in \mathrm{End}_{\mathbb{C}}(M \otimes V^{\otimes d}$. Then,

Theorem 1.2.4. *(Arakawa-Suzuki) There exists a homomorphism*

$$\mathcal{H}^{\mathrm{aff}}(d) \to \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d}), \quad x_i \mapsto X_i, \ s_i \mapsto S_{i,i+1}.$$

Conjecture 1.2.5. *This map is surjective provided $n \geq d$.*

In order to generalize this result, one should really think about this action coming from the *bi-algebra structure on* $\mathfrak{g}$. To describe this, let

$$r^+ = \frac{1}{2}\sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{1 \leq i < j \leq n} e_{ij} \otimes e_{ji}$$

be the classical r-matrix. This matrix satisfies

$$\Delta(X) = [1 \otimes X + X \otimes 1, r^+]$$

for all $X \in \mathfrak{g}$, where $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is comultiplication. The r-matrix satisfies the classical Yang-Baxter equation

$$[r^+_{12}, r^+_{13}] + [r^+_{12}, r^+_{23}] + [r^+_{13}, r^+_{23}] = 0$$

giving Category $\mathcal{O}$ for $\mathfrak{g}$ the structure of a *braided category.*

Note that if we take

$$r^- = \frac{1}{2}\sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{1 \le j < i \le n} e_{ij} \otimes e_{ji}$$

then $\Omega = r^+ \oplus r^-$ is the Casimir. In [**?**], Orellana and Ram explain this phenomenon in type A. There are many interesting applications.

Exercise 1.2.6. *Give some content to the previous discussion.*

We have now defined the homomorphism (1). Let $\lambda \in P$ be a weight and M be a module in category $\mathcal{O}$. Since the action of $\mathcal{H}^{\mathrm{aff}}(d)$ on $M \otimes V^{\otimes d}$ commutes with that of $\mathfrak{g}$, it follows that $\mathcal{H}^{\mathrm{aff}}(d)$ acts on the set of primitive vectors in $(M \otimes V^{\otimes d})^{[\lambda]}$ of weight λ. That is, the space

$$F_\lambda M = \{m \in (M \otimes V^{\otimes d})_\lambda | \mathfrak{n}^+ . m = 0\}.$$

has the structure of an $\mathcal{H}^{\mathrm{aff}}(d)$-module, which is finite dimensional. This means that for each $\lambda \in P$, we have a functor

$$F_\lambda : \mathcal{O} \to \mathcal{H}^{\mathrm{aff}}(d)\text{-mod}.$$

Exercise 1.2.7. *Show that there are natural isomorphisms*

$$F_\lambda M \cong \mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M \otimes V^{\otimes d}) \cong \left(\frac{M \otimes V^{\otimes d}}{\mathfrak{n}^-(M \otimes V^{\otimes d})} \right)^{[\lambda]}_{\lambda}.$$

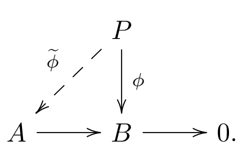
Using the isomorphisms of the previous exercise, we prove

Proposition 1.2.8. *If $\lambda + \rho \in P^+$, then F_λ is exact.*

Proof: We will use the Hom description of the functor. Indeed, F_λ is the composition

$$F_\lambda(?) \cong \mathrm{Hom}_{\mathfrak{g}}(M(\lambda), ? \otimes V^{\otimes d}) = \mathrm{Hom}_{\mathfrak{g}}(M(\lambda), ?) \circ (? \otimes V^{\otimes d}).$$

Tensoring with ia finite dimensional vector space is exact. For an abelian category $\mathcal{C}$, the functor $\mathrm{Hom}_{\mathcal{C}}(P, ?)$ is exact precisely when P is projective. It is always left exact. To show that it is right exact, we need to show that given a surjection $A \to B \to 0$ and $\phi \in \mathrm{Hom}_{\mathcal{C}}(P, B)$, there exists an element $\widetilde{\phi} \in \mathrm{Hom}_{\mathcal{C}}(P, A)$ such that the following diagram commutes:



This can always be done if and only if P is projective. Finally, a Verma module $M(\lambda)$ is projective if, and only if, $\lambda + \rho \in P^+$. ∎

Verma modules form a basis for the Grothendieck group $K(\mathcal{O})$. It is therefore important to understand their image under the functor. The following proposition gives an effective way to compute $\dim F_\lambda M(\mu)$, and describes its structure as a $\mathbb{C}S_d$-module.

Proposition 1.2.9. *The natural embedding $(V^{\otimes d})_{\lambda-\mu} \hookrightarrow F_\lambda M(\mu)$ given by $u \mapsto v_\mu \otimes u$ is an isomorphism of $\mathbb{C}S_d$-modules. In particular, $F_\lambda M(\mu) = 0$ unless $\lambda - \mu \in P(V^{\otimes d})$.*

Proof: By the tensor identity and the PBW theorem,

$$M(\mu) \otimes V^{\otimes d} \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{n}^+ \oplus \mathfrak{h})} \left(\mathbb{C}v_\mu \otimes V^{\otimes d}\right) \cong U(\mathfrak{n}^-) \otimes \mathbb{C}v_\mu \otimes V^{\otimes d}, \tag{2}$$

where the first isomorphism is as $\mathfrak{g}$-modules and the second is as $\mathfrak{h}$-modules. Thus the canonical projection map induces the isomorphism of $\mathfrak{h}$-modules given by

$$1 \otimes \mathbb{C}v_\mu \otimes V^{\otimes d} \cong M(\mu) \otimes V^{\otimes d} / \mathfrak{n}^- \left(M(\mu) \otimes V^{\otimes d}\right).$$

Taking λ weight spaces on both sides yields the vector space isomorphism

$$1 \otimes \mathbb{C}v_\mu \otimes \left(V^{\otimes d}\right)_{\lambda-\mu} \cong \left[M(\mu) \otimes V^{\otimes d} / \mathfrak{n}^- \left(M(\mu) \otimes V^{\otimes d}\right)\right]_\lambda.$$

Now, the composition of the natural inclusion $\mathbb{C}v_\mu \otimes (V^{\otimes d})_{\lambda-\mu} \hookrightarrow (M(\mu) \otimes V^{\otimes d})_\lambda$ with (2), and the isomorphism above implies that

$$\mathbb{C}v_\mu \otimes \left(V^{\otimes d}\right)_{\lambda-\mu} \cong 1 \otimes \mathbb{C}v_\mu \otimes \left(V^{\otimes d}\right)_{\lambda-\mu} \cong \left[M(\mu) \otimes V^{\otimes d} / \mathfrak{n}^- \left(M(\mu) \otimes V^{\otimes d}\right)\right]_\lambda = F_\lambda\left(M(\mu)\right),$$

That it is an isomorphism of $\mathbb{C}S_d$-modules follows from the fact that in each case the action of $\mathbb{C}S_d$ is via the action induced from the action of $\mathbb{C}S_d$ on $M(\mu) \otimes V^{\otimes d}$. ∎

From now on, fix $\lambda \in P^+$ and $\mu \in \lambda - P(V^{\otimes d})$. We want to understand the action of $x_1, \dots, x_d$ on $F_\lambda M(\mu)$. To this end, we fix the following notation:

- Let $\lambda - \mu = (d_1, \dots, d_n)$, so $d_1 + \cdots + d_n = d$ and $d_i \geq 0$ for all i.
- Let $m_k = \sum_{i=1}^k d_i$, and $m_0 = 0$.
- Let $u_{\lambda-\mu} = u_1^{\otimes d_1} \otimes \cdots \otimes u_n^{\otimes d_n}$, where $\{u_1, \dots, u_n\}$ is the standard (coordinate) basis for V.

We make the following useful observations:

(1) $\mathfrak{n}^-(M(\mu) \otimes V^{\otimes d}) = 0$ implies $(Ev_\mu) \otimes u_{\lambda-\mu} = v_\mu \otimes Eu_{\lambda-\mu}$ for all $E \in U(\mathfrak{n}^-)$.

(2) If $m_{k-1} < i < j \leq m_k$, then $S_{ij}u_{\lambda-\mu} = u_{\lambda-\mu}$.

Lemma 1.2.10. $\dim F_\lambda M(\mu) = \frac{d!}{d_1! \cdots d_n!}$

Proof: By the previous proposition, $\dim F_\lambda M(\mu) = \dim(V^{\otimes d})_\lambda - \mu$. It is easy to see that $(V^{\otimes d})_\lambda - \mu = \mathbb{C}S_d.u_{\lambda-\mu}$ and by observation (2) above $\mathbb{C}S_d.u_{\lambda-\mu} \cong \mathbb{C}[S_d/S_{\lambda-\mu}]$. Hence the result. ∎

Lemma 1.2.11. *For each $1 \leq k \leq n$ and $m_{k-1} < i \leq m_k$*

$$X_i(v_\mu \otimes u_{\lambda-\mu}) = (\mu_k + i - m_{k-1} - k)v_\mu \otimes u_{\lambda-\mu}.$$

Exercise 1.2.12. *Recall that $X_i = \Omega_{0i} + \sum_{j<i} S_{ij}$. We may decompose $\Omega_{0i} = r_{0i}^+ + r_{0i}^-$. Using the observations (1) and (2) above, compute*

(i) $r_{0i}^+.v_\mu \otimes u_{\lambda-\mu} = \frac{1}{2}\mu_k.v_\mu \otimes u_{\lambda-\mu}$;

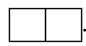
(ii) $r_{0,i}^-.v_\mu \otimes u_{\lambda-\mu} = \left(\frac{1}{2}\mu_k - \sum_{1 \leq j \leq m_{k-1}} S_{ij} - k + 1\right).v_\mu \otimes u_{\lambda-\mu}$;

(iii) $\sum_{1\leq j<i} S_{ij}.v_\mu \otimes u_{\lambda-\mu} = \left(\sum_{1\leq j\leq m_{k-1}} S_{ij} + i - m_{k-1} - 1\right).v_\mu \otimes u_{\lambda-\mu}$.
Deduce the lemma by adding together (i)+(ii)+(iii).

Example 1.2.13. *We compute some examples for $n = d = 2$. To this end, let $\lambda \in P^+$, and $\mu = (\mu_1, \mu_2)$. Observe that*

$$\Omega = e_{11} \otimes e_{11} + e_{22} \otimes e_{22} + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}.$$

(A) $\lambda - \mu = (2, 0)$. Then, $u_{\lambda-\mu} = u_1 \otimes u_1$, and we are considering the partition

.

We calculate

1. $X_1(v_\mu \otimes u_1 \otimes u_1) = \mu_1 v_\mu \otimes u_1 \otimes u_1$.
2. *To calculate $X_2(v_\mu \otimes u_1 \otimes u_1)$ we calculate*

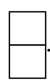
$$\Omega_{02}(v_\mu \otimes u_1 \otimes u_1) = \mu_1 v_\mu \otimes u_1 \otimes u_1,$$

and

$$\Omega_{12}(v_\mu \otimes u_1 \otimes u_1) = v_\mu \otimes u_1 \otimes u_1.$$

Therefore, $X_2(v_\mu \otimes u_1 \otimes u_1) = (\Omega_{02} + \Omega_{12})(v_\mu \otimes u_1 \otimes u_1) = (\mu_1 + 1)v_\mu \otimes u_1 \otimes u_1$.

(B) $\lambda - \mu = (1, 1)$. Then $u_{\lambda-\mu} = u_1 \otimes u_2$ and we are considering the partition

.

We calculate

1. $X_1(v_\mu \otimes u_1 \otimes u_2) = \mu_1 v_\mu \otimes u_1 \otimes u_2$.
2. *To calculate $X_2(v_\mu \otimes u_1 \otimes u_2)$ we calculate*

$$\begin{aligned}\Omega_{02}(v_\mu \otimes u_1 \otimes u_2) &= \mu_2 v_\mu \otimes u_1 \otimes u_2 + (e_{21} v_\mu) \otimes u_1 \otimes u_1 \\ &= \mu_2 v_\mu \otimes u_1 \otimes u_2 - v_\mu \otimes (e_{21}u_1) \otimes u_1 - v_\mu \otimes u_1 \otimes (e_{21}u_1) \\ &= \mu_2 v_\mu \otimes u_1 \otimes u_2 - v_\mu \otimes u_2 \otimes u_1 - v_\mu \otimes u_1 \otimes u_2\end{aligned}$$

and

$$\Omega_{12}(v_\mu \otimes u_1 \otimes u_2) = v_\mu \otimes u_2 \otimes u_1.$$

Therefore,

$$\begin{aligned}X_2(v_\mu \otimes u_1 \otimes u_2) &= (\Omega_{02} + \Omega_{12})(v_\mu \otimes u_1 \otimes u_2) \\ &= (\mu_2 v_\mu \otimes u_1 \otimes u_2 - v_\mu \otimes u_2 \otimes u_1 - v_\mu \otimes u_1 \otimes u_2) + (v_\mu \otimes u_2 \otimes u_1) \\ &= (\mu_2 - 1)v_\mu \otimes u_1 \otimes u_2.\end{aligned}$$

A segment is an interval $[a, b] = \{a, a + 1, \ldots, b - 1, b\} \subseteq \mathbb{Z}$. Given a segment $[a, b]$, with $d = b - a + 1 \geq 0$, associate a 1-dimansional H_d-module, $\mathbb{C}\mathbf{1}_{[a,b]}$ defined by $s_i \mathbf{1}_{[a,b]} = \mathbf{1}_{[a,b]}$ and $x_i \mathbf{1}_{[a,b]} = (a + i - 1)\mathbf{1}_{[a,b]}$. (When $d = 0$, we interpret $[a, a - 1] = \emptyset$, and $\mathbb{C}\mathbf{1}_\emptyset$ is a 1-dimensional H_0-module = $\mathbb{C}$-vector space.)

Exercise 1.2.14. *Show that $\mathbb{C}\mathbf{1}_{[a,b]}$ is, in fact, an H_d-module.*

A multisegment $\Delta = ([a_1, b_1], \ldots, [a_n, b_n])$ is a (finite) sequence of segments. Given a multisegment, Δ as above, with $b_i - a_i + 1 = d_i$ and $d = d_1 + \cdots + d_n$, we define the *standard cyclic* module

$$\begin{aligned}\mathcal{M}(\Delta) &= \mathrm{Ind}^d_{d_1,\ldots,d_n} \mathbb{C}\mathbf{1}_{[a_1,b_1]} \boxtimes \cdots \boxtimes \mathbb{C}\mathbf{1}_{[a_n,b_n]} \\ &= H_d \otimes_{H_{d_1}\otimes \cdots H_{d_n}} \mathbb{C}\mathbf{1}_{[a_1,b_1]} \boxtimes \cdots \boxtimes \mathbb{C}\mathbf{1}_{[a_n,b_n]}.\end{aligned}$$

Remark 1.2.15. *Let $\nu = (d_1, \ldots, d_n)$ be the composition defined by the integers $d_1, \ldots, d_n$ above and $\nu^+ = (d_{i_1} \geq \cdots \geq d_{i_n})$ be the associated partition. Then,*

$$\mathcal{M}(\Delta) \downarrow_{\mathbb{C}S_d} \cong \mathcal{M}^{\nu^+}$$

where $\mathcal{M}^{\nu^+}$ is the permutation module defined at the beginning of the term.

Exercise 1.2.16. *Show that if $\mathcal{M}(\Delta)$ is as above, then*

$$\dim \mathcal{M}(\Delta) = \frac{d!}{d_1! \cdots d_n!}.$$

Theorem 1.2.17. *Given $\lambda \in P^+ - \rho$ and $\mu \in \lambda - P(V^{\otimes d})$, associate the multisegment $\Delta = ([\mu_1, \lambda_1 - 1], [\mu_2 - 1, \lambda_2 - 2], \ldots, [\mu_n - n + 1, \lambda_n - n])$ and the standard cyclic module*

$$\mathcal{M}(\lambda, \mu) := \mathcal{M}(\Delta) = \mathrm{Ind}^d_{\lambda-\mu} \mathbb{C}\mathbf{1}_{[\mu_1,\lambda_1-1]} \boxtimes \cdots \boxtimes \mathbb{C}\mathbf{1}_{[\mu_n-n+1,\lambda_n-n]}.$$

Then, $F_\lambda M(\mu) \cong \mathcal{M}(\lambda, \mu)$.

Proof: Let $\mathbf{1}_{\lambda,\mu} = \mathbf{1}_{[\mu_1,\lambda_1-1]} \otimes \cdots \otimes \mathbf{1}_{[\mu_n-n+1,\lambda_n-n]}$. There is an injective $H_{\lambda-\mu}$-homomorphism $\mathbb{C}\mathbf{1}_{\lambda,\mu} \to F_\lambda M(\mu)$ given by $\mathbf{1}_{\lambda,\mu} \mapsto v_\mu \otimes u_{\lambda-\mu}$. Frobenius reciprocity implies that there exists an injective homomorphism $\mathcal{M}(\lambda, \mu) \to F_\lambda M(\mu)$. Now the result follows because these modules have the same dimension. ∎

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