REPRESENTATION THEORY OF SYMMETRIC GROUPS AND HECKE ALGEBRAS

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1. Symmetric Groups

1.1. **Representations of Groups.** In this section, we review some basic facts from the representation theory of finite groups. To this end, let F be a field of characteristic $p \geq 0$ and let G be a finite group.

Theorem 1.1.1. *If M is an irreducible FG module, then M is a composition factor of the group algebra FG.*

Proof: Let $m \in M$ be nonzero. Then, $FG.m \leq M$ is a nonzero submodule, hence equals M. The map

$$\theta :_{FG} FG \to M, \quad x \mapsto x.m$$

is an FG-homomorphism. By the first isomorphism theorem, ${}_{FG}FG/\ker\theta \cong M$, so M is a composition factor of FG. ∎

Recall that an FG-module, M, is called *completely reducible* if whenever $X \leq M$ is a submodule, there exists a submodule $Y \leq M$ such that $M \cong X \oplus Y$. A nonzero FG-module is called *simple* or *irreducible* if it has no proper nontrivial submodules. The group algebra FG is called *semisimple* if every FG-module is completely reducible.

Theorem 1.1.2. *(Maschke's Theorem) If $\mathrm{Char} F \nmid |G|$, then FG is semisimple.*

Proof: Assume that M is an FG module and $X \leq M$ is a submodule. Let $Y \subseteq M$ be the *linear* compliment of X, i.e. $M \cong X \oplus Y$ *as vector spaces.* The projection $\pi : M \to X$ is a linear map. Define the map

$$\varphi : M \to M,$$

by

$$\varphi(m) = |G|^{-1} \sum_{g \in G} g^{-1}\pi(g.m)$$

for all $m \in M$. Observe that this formula makes sense since $|G| \neq 0 \in F$. To see this is an FG-homomorphism, observe that if $h \in G$ and $m \in M$,

$$\begin{aligned}
\varphi(h.m) &= |G|^{-1} \sum_{g \in G} g^{-1}\pi(g.(h.m)) \\
&= |G|^{-1} \sum_{g \in G} hh^{-1}g^{-1}\pi((gh).m) \\
&= h|G|^{-1} \sum_{g \in G} (gh)^{-1}\pi((gh).m) \\
&= h\varphi(m).
\end{aligned}$$

Now, we show that $\operatorname{im}\varphi = X$. Indeed, $\pi(x) = x$ for all $x \in X$, and X is a submodule, so

$$\begin{aligned}\varphi(x) &= |G|^{-1}\sum_{g\in G} g^{-1}\pi(g.x)\\ &= |G|^{-1}\sum_{g\in G} g^{-1}gx\\ &= |G|^{-1}|G|x = x.\end{aligned}$$

Now, by standard arguments, $M \cong X \oplus \ker\varphi$. ∎

Exercise 1.1.3. *Show that if* Char$F \mid |G|$*, then the 1-dimensional submodule of* ${}_{FG}FG$ *spanned by* $\sum_{g\in G} g$ *is not a direct summand.*

Exercise 1.1.4. *Assume that the* Char$F \nmid |G|$*. If M is any FG-module, Show that M admits a nonzero, G-invariant, symmetric bilinear form.*

Exercise 1.1.5. *Show that if $\langle\cdot,\cdot\rangle$ is a G-invariant bilinear form and $X \le M$ is a submodule, then* $X^\perp = \{y \in M | \langle x, y\rangle = 0,\ \text{for all } x \in X\}$ *is a submodule.*

For the remainder of this section, we specialize to the case $F = \mathbb{C}$. The group algebra $\mathbb{C}G$ is semisimple, and applying the powerful *Wedderburn-Artin Theorem* we have that

Theorem 1.1.6. *There exist $n_1, \dots, n_m \in \mathbb{Z}_{\ge 0}$ such that*

$$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_m}(\mathbb{C}).$$

In particular,

$$|G| = n_1^2 + \cdots + n_m^2.$$

Let $C_1, \dots, C_r$ be the conjugacy classes in a finite group G. For each C_j, define the element

$$z_j = \sum_{g\in C_j} g.$$

The elements $z_1, \dots, z_r$ are called *class sums.*

Theorem 1.1.7. *If r is the number of conjugacy classes in G, then*

$$r = \dim_{\mathbb{C}}(Z(\mathbb{C}G)),$$

where $Z(\mathbb{C}G)$ is the center of the group algebra. In fact, a basis of $Z(\mathbb{C}G)$ consists of all class sums.

Proof: First of all, if $h \in G$, then

$$hz_j = \sum_{g\in C_j} hg = \sum_{g\in C_j}(hgh^{-1})h = z_jh$$

so $z_j \in Z(\mathbb{C}G)$. Also, note that if $j \ne \ell$, then z_j and z_ℓ are sums of disjoint sets of group elements. Hence, $\{z_1, \dots, z_r\}$ is a linearly independent set.

We are left to show that the z_j span $Z(\mathbb{C}G)$. To this end, let $u \in Z(\mathbb{C}G)$, and write

$$u = \sum_{g\in G} a_g g.$$

Then, for all $h \in G$,

$$u = huh^{-1} = \sum_{g \in G} a_g hgh^{-1} = \sum_{g \in G} a_{h^{-1}gh} g.$$

It follows that $a_{hgh^{-1}} = a_g$ for all $g \in G$. That is, $a_g = a_{g'} = a_j$ if $g, g' \in C_j$. Hence,

$$u = \sum_{j=1}^{r} a_j z_j.$$

∎

Combining the previous two theorems, we obtain that

Theorem 1.1.8. *The number, m, of simple components in $\mathbb{C}G$ is equal to the number r of conjugacy classes in G.*

Proof: We know that $r = \dim_{\mathbb{C}} Z(\mathbb{C}G)$. On the over hand, $Z(M_{n_i}(\mathbb{C}))$ is 1-dimensional consisting of scalar matrices, so

$$r = \dim_{\mathbb{C}} Z(\mathbb{C}G) = \dim_{\mathbb{C}} Z(M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_m}(\mathbb{C})) = m.$$

∎

Corollary 1.1.9. *The number of irreducible complex representations of $\mathbb{C}G$ equals the number of conjugacy classes of G.*

Proof: The matrix algebra $M_{n_i}(\mathbb{C})$ has exactly one irreducible representation up to isomorphism, namely C^{n_i}. So

$$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_m}(\mathbb{C})$$

Has exactly m irreducible representations up to isomorphism. By the previous theorem, this equals the number of conjugacy classes. ∎

1.2. **Symmetric Groups.** The Symmetric group S_n is the group of bijections of the set $\{1, \dots, n\}$ with multiplication given by composition of functions. It is common to write a permutation $\sigma \in S_n$ as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix}.$$

By considering orbits of the elements $\{1, \dots, n\}$ with respect to the action of σ, it is obvious that σ can be written as a product of disjoint cycles. For example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 6 & 5 & 4 & 1 & 7 \end{pmatrix} = (1236)(45)(7).$$

We will follow the usual convention and omit 1-cycles.

Exercise 1.2.1. *(a) Show that every permutation $\sigma \in S_n$ can be written as a product of transpositions.*

(b) Show that if $\sigma = \tau_1 \cdots \tau_m = \delta_1 \cdots \delta_k$, then $m \equiv k$ (mod 2).

(c) Conclude that there is a well defined homomorphism $\operatorname{sgn} : S_n \to \mathbb{Z}^\times = \{\pm 1\}$.

Theorem 1.2.2. *The symmetric group is generated by the* simple *transpositions $s_i = (i, i+1)$ for $1 \le i < n$.*

Proof: By the previous exercise 1.2.1, S_n is generated by transpositions. We need only show that each transpositions (a,b) (say, $a<b$) can be written as a product of simple transpositions.

To this end, proceed by induction on $b-a$. If $b-a=1$, then (a,b) is a simple transposition. If $b-a>1$, then

$$(a,b)=(a,a+1)(a+1,b)(a,a+1).$$

Since $b-(a+1)<b-a$, induction implies that $(a+1,b)$ can be written as a product of simple transpositions. ∎

Definition 1.2.3. *A partition of n is a sequence $\lambda=(\lambda_1,\lambda_2,\ldots)$ such that $\lambda_i\geq\lambda_{i+1}$ for all i and $\sum_i\lambda_i=n$. Let $P(n)$ denote the set of partitions of n. When λ is a partition of n, we sometimes write $\lambda\vdash n$ instead of $\lambda\in P(n)$.*

Let $\sigma=\rho_1\cdots\rho_m$ be the decomposition of σ as a product of disjoint cycles including 1-cycles, and assume that ρ_i is a cycle of length λ_i. Since disjoint cycles commute, we may assume that $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_m$. In this case, we say that σ as cycle type *$\lambda=(\lambda_1,\ldots,\lambda_m,0,0,\ldots)$.*

Exercise 1.2.4. *(a) Let $\rho=(1,\ldots,k)\in S_n$. Show that if $\sigma\in S_n$, then*

$$\sigma\rho\sigma^{-1}=(\sigma(1),\ldots,\sigma(k)).$$

(b) Deduce that two permutations σ and τ are conjugate if, and only if, they have the same cycle type.

A corollary of this exercise we deduce that

Corollary 1.2.5. *The number of irreducible representations of $\mathbb{C}S_n$ is equal to the number of partitions of n.*

Exercise 1.2.6. *This exercise is devoted to constructing the most obvious representation of S_n.*

(1) Let $V=F^n$ be an n-dimensional F-vector space, and let $e_1,\ldots,e_n$ be the coordinate basis for V. That is, e_i is a column vector with 1 in the ith row and 0 elsewhere. Show that there is an injective homomorphism $FS_n\to M_n(C)$ given by $\sigma\mapsto A_\sigma$, where $A_\sigma(e_i)=e_{\sigma(i)}$.

(2) Show that the subspace spanned by $e_1+\cdots+e_n$ is a FS_n submodule. Prove that if $p\nmid n!$, then the complimentary subspace $I=\{\sum a_ie_i|\sum a_i=0\}$ is also a submodule.

1.3. **Bilinear Forms.** In the previous sections, we determined that the conjugacy classes for the symmetric group S_n are labeled by partitions of n and that this is the number of irreducible complex representations of $\mathbb{C}S_n$. For a more general field F, this gives an upper bound on the number of irreducible representations of FS_n.

Motivated by this, we will associate to each partition λ a so call *permutation module* M^λ for the group algebra FS_n (F is now *any* field!). This module is not irreducible but always admits a nondegenerate, S_n-invariant, symmetric bilinear form.

We identify an interesting submodule $S^\lambda\leq M^\lambda$ known as a Specht module. This module may or may not be irreducible depending on the characteristic of the field. Let $(S^\lambda)^\perp$ denote the orthogonal compliment of S^λ with respect to the S_n-invariant form on M^λ. The main theorem we would like to prove about S^λ is the following:

Theorem. *The module $D^\lambda = S^\lambda/(S^\lambda \cap (S^\lambda)^\perp)$ is either 0 or absolutely irreducible. When it is nonzero, $S^\lambda \cap (S^\lambda)^\perp$ is the unique maximal submodule of S^λ, and D^λ is self dual.*

In this section, we explain why D^λ is self-dual using general facts about bilinear forms. To this end, let F be any field and G be a finite group.

Let M be a finite dimensional vector space over F, and let

$$M^* = \operatorname{Hom}_F(M, F).$$

Let $V \subseteq M$ be a subspace of M and $e_1, \dots, e_k$ be a basis for V. Extend this to a basis $e_1, \dots, e_m$ for M. For $j = 1, \dots, m$, let $\varepsilon_j \in M^*$ be defined by $\varepsilon_j(e_i) = \delta_{ij}$, where δ_{ij} is the Kronecker δ. Obviously, $\varepsilon_1, \dots, \varepsilon_m$ forms a basis for M^* and hence,

$$\dim_F M = \dim_F M^*.$$

Let $V^\circ \subseteq M^*$ denote the annihilator of V. Clearly, V° has basis $\varepsilon_{k+1}, \dots, \varepsilon_m$, so

$$\dim_F V + \dim_F V^\circ = \dim_F M.$$

Now, suppose that M possesses a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. Define a map

$$\theta : M \to M^*, \quad \theta(m) = \psi_m$$

where $\psi_m(x) = \langle m, x \rangle$. Note that $\psi_m \in M^*$ because $\langle \cdot, \cdot \rangle$ is linear in the second place, and θ is a linear map because $\langle \cdot, \cdot \rangle$ is linear in the first place. Moreover, $\ker \theta = 0$ since the form is nondegenerate. Since $\dim_F M = \dim_F M^*$, θ is an isomorphism, and $\theta(V^\perp) = V^\circ$. In particular, for any subspace V,

$$\dim_F V + \dim_F V^\perp = \dim_F M.$$

Also, we obviously have $V^{\perp\perp} = V$.

More generally, assume we have $0 \subseteq U \subseteq V \subseteq M$. Then $U^\perp \supseteq V^\perp$ and we may define a map

$$\Theta : V \to (U^\perp/V^\perp)^*$$

by $\Theta(v) = \psi_v$, where $\psi_v(x + V^\perp) = \langle v, x \rangle$.

Exercise 1.3.1. *Prove the Θ is well-defined.*

As before, Θ is linear, but now

$$\ker \Theta = \{v \in V | \langle v, x \rangle = 0 \text{ for all } x \in U^\perp\} = V \cap U^{\perp\perp}.$$

Now, $U^{\perp\perp} = U \subseteq V$, so $\ker \theta = U$. Applying the first isomorphism theorem we have

Proposition 1.3.2. *When $0 \subseteq U \subseteq V \subseteq M$, $V/U \cong (U^\perp/V^\perp)^*$.*

Now, assume M is additionally an FG-module. We may define an FG-module structure on M^* by $(g.\psi)(m) = \psi(g^{-1}.m)$. Note that in general, M is not isomorphic to M^* as FG-modules, however this is true if the bilinear form on M is G-invariant.

Indeed, assume that the bilinear form on M is G-invariant. We will show that under this assumption the map Θ above becomes and FG-homomorphism. (In the special case $U = 0$ and $V = M$ this will prove $M \cong M^*$.)

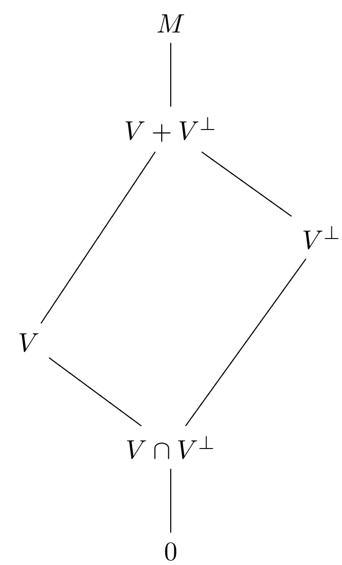
Now, for $g \in G$, $v \in V$ and $x + V^{\perp} \in U^{\perp}/V^{\perp}$, we have

$$\psi_{g.v}(x + V^{\perp}) = \langle g.v, x\rangle = \langle g^{-1}g.v, g^{-1}.x\rangle = \langle v, g^{-1}.x\rangle = \psi_v(g^{-1}.(x + V^{\perp})) = (g.\psi_v)(x + V^{\perp}).$$

Hence, $\Theta(gv) = g\Theta(v)$ as required.

Now, for every pair of subspaces $U, V \subseteq M$, we have $(U+V)^{\perp} = U^{\perp} \cap V^{\perp}$, and (taking orthogonal compliments) $U^{\perp} + V^{\perp} = (U \cap V)^{\perp}$.

Next, consider the following picture:



By the previous paragraph, the second isomorphism theorem and Proposition 1.3.2 it follows that

$$\begin{aligned} V/(V \cap V^{\perp}) &\cong (V + V^{\perp})/V^{\perp} \\ &\cong (V/(V + V^{\perp})^{\perp})^{*} = (V/V \cap V^{\perp})^{*}. \end{aligned}$$

Hence,

Theorem 1.3.3. *If M is an FG-module equipped with a nondegenerate, G-invariant, symmetric bilinear form, then, for every FG-submodule $V \leq M$, $V/V \cap V^{\perp}$ is self-dual.*

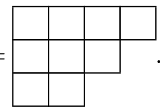
The next exercise explains how to compute $\dim V/V \cap V^{\perp}$ in principal.

Exercise 1.3.4. *Recall that $e_1, \ldots, e_k$ is a basis for V. Let $A = (\langle e_i, e_j\rangle)_{i,j=1}^{k}$ be the Gram matrix of the bilinear form $\langle \cdot, \cdot\rangle$ on V with respect to this basis. Prove that*

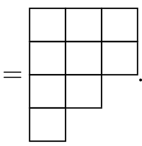
$$\dim V/V \cap V^{\perp} = \operatorname{rank}(A).$$

1.4. **Partitions, Tableaux and Tabloids.** The set $P(n)$ is both partially ordered and totally ordered. The standard partial ordering on partitions, called the *dominance ordering* is defined by $\lambda \lhd \mu$ if $\lambda_1 + \cdots + \lambda_i < mu_1 + \cdots + \mu_i$ for all i. The total ordering on partitions is called the *lexicographic* or *dictionary* ordering is defined by $\lambda < \mu$ if $\lambda_j < \mu_j$ and $\lambda_i = \mu_i$ for all $i < j$. Note that in some papers (especially those concerning symmetric functions), the lexicographic ordering is reversed.

To each partition λ, we associate a *Young diagram* denoted $[\lambda]$. A Young diagram is a left-justified array of boxes, having λ_1 boxes in the first row, λ_2 boxes in the second, and so on. For example, if $\lambda = (4, 3, 2)$,

$[\lambda] =$  .

Given a partition $\lambda \vdash n$, its transpose λ^t is the partition obtained by swapping the rows and columns of $[\lambda]$. For example, for the partition $\lambda = (4, 3, 2)$, its transpose is $\lambda^t = (3, 3, 2, 1)$:

$[\lambda^t] =$  .

A λ-*tableaux* is a filling of the diagram $[\lambda]$ with the integers $1, \dots, n$, each occurring exactly once. My favorite filling is the standard row filling. For example, if $\lambda = (4, 3, 2)$ as above, the standard row filling of $[\lambda]$ is the tableaux

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 \\ \cline{1-2} \end{array} .$$

The following fundamental result relates the dominance ordering on partitions to tableaux:

Lemma 1.4.1. *Assume $\lambda, \mu \vdash n$, T_1 is a λ-tableaux and T_2 is a μ-tableaux. Assume that for each i, the numbers in the ith row of T_2 belong to different columns of T_1. Then $\lambda \trianglerighteq \mu$.*

Proof: If we can place the μ_1 entries in the first row of T_1 in different columns of $[\lambda]$, then $[\lambda]$ must have at least μ_1 columns. That is, $\lambda_1 \geq \mu_1$. Now, assuming we have done this, we can place the μ_2 entries in T_2 in distinct columns of $[\lambda]$ only if there are μ_2 distinct free columns in λ remaining. That is, we need $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$. Continuing, we deduce that $\lambda \trianglerighteq \mu$. ∎

Now, the symmetric group acts on the set of λ-tabloids by place permutation. For example,

$$s_1 \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} .$$

Given a λ-tableaux T, define its *row stabilizer*,

$$R(T) = \{w \in S_n |\text{ for all } i \text{ both } i \text{ and } w(i) \text{ belong to the same row of } T\}.$$

In other words, an element w belongs to $R(T)$ if, and only if, w fixes the rows of T set-wise. The row stabilizer is isomorphic to a *Young subgroup*, $S_\lambda \cong S_{\lambda_1} \times \cdots \times S_{\lambda_m}$, of S_n. If T is a standard row filling of $[\lambda]$, then

$$R(T) = S_\lambda := S_{\{1,\dots,\lambda_1\}} \times S_{\{\lambda_1+1,\dots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{\lambda_1+\cdots+\lambda_{m-1}+1,\dots,n\}}. \qquad (1)$$

There is a similar definition for the *column stabilizer*, $C(T)$.

Exercise 1.4.2. *Prove that for $\sigma \in S_n$, $\sigma R(T)\sigma^{-1} = R(\sigma T)$ and $\sigma C(T)\sigma^{-1} = C(\sigma T)$.*

Define a λ-tabloid, $\{T\}$ to be the $R(T)$ orbit of a λ-tableaux T: $\{T\} = R(T)T$. That is, a λ-tabloid is a λ-tableaux with unordered rows. For example,

$$\{T\} = \left\{ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 \\ \cline{1-2} \end{array} \right\} = \left\{ \begin{array}{|c|c|c|c|} \hline 2 & 3 & 1 & 4 \\ \hline 5 & 7 & 6 \\ \cline{1-3} 8 & 9 \\ \cline{1-2} \end{array} \right\}$$

1.5. **Permutation Modules and Specht Modules.** As discussed in the previous section, S_n acts on the set of tableaux. Obviously, this action is transitive. As a consequence, S_n acts transitively on the set of tabloids via

$$\sigma\{T\} = \{\sigma T\}.$$

Therefore, given a partition $\lambda \vdash n$, it makes sense to define the *permutation module*

$$M^\lambda = \bigoplus_{\{T\}} F\{T\}.$$

Again, since the action of S_n on tabloids is transitive, we have

Proposition 1.5.1. *The permutation module M^λ is cyclically generated by any λ-tabloid $\{T\}$:*

$$M^\lambda = FS_n.\{T\}$$

.

Exercise 1.5.2. *Show that the module $V = F^n$ is constructed in exercise 1.2.6 is isomorphic to $M^{(n-1,1)}$.*

Corollary 1.5.3. *$M^\lambda \cong FS_n \otimes_{FS_\lambda} \mathbf{1}_\lambda$, where $\mathbf{1}_\lambda$ is the 1-dimensional trivial FS_λ-module.*

Proof: Frobenius reciprocity says that if $H \leq G$, then

$$\mathrm{Hom}_{FG}(FG \otimes_{FH} V, W) = \mathrm{Hom}_{FH}(V, W).$$

We now apply this to our situation. There exists an injective homomorphism $\theta \in \mathrm{Hom}_{FR(T)}(\mathbf{1}_\lambda, M^\lambda)$ so that $\theta(\mathbf{1}_\lambda) = \{T\}$. This corresponds to an injective map

$$\tilde{\theta} \in \mathrm{Hom}_{FS_n}(FS_n \otimes_{FR(T)} \mathbf{1}_\lambda, M^\lambda),$$

which is an isomorphism by comparing dimensions. ∎

Exercise 1.5.4. *Suppose $\lambda = (\lambda_1, \ldots, \lambda_k)$, and T is a λ-tableaux. Prove that $|R(T)| = \lambda_1! \cdots \lambda_k!$.*

Given a λ-tableaux T, define the *row sum*

$$\rho_T = \sum_{\sigma \in R(T)} \sigma$$

and the *signed column sum*

$$\kappa_T = \sum_{\sigma \in C(T)} \mathrm{sgn}(\sigma)\sigma.$$

Exercise 1.5.5. *Show that $\rho_T^2 = |R(T)|\rho_T$ and $\kappa_T^2 = |C(T)|\kappa_T$.*

We are now prepared to define Specht modules. To this end, given a λ-tableaux T, define a *polytabloid* e_T by the formula

$$e_T = \kappa_T\{T\}.$$

For example, if $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$, then

$$\kappa_T = 1 - (14) - (25) + (14)(25) = (1-(14))(1-(25))$$

and

$$e_T = \left\{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 4 & 2 & 3 \\ \hline 1 & 5 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 1 & 5 & 3 \\ \hline 4 & 2 \\ \cline{1-2}\end{array}\right\} + \left\{\begin{array}{|c|c|c|}\hline 4 & 5 & 3 \\ \hline 1 & 3 \\ \cline{1-2}\end{array}\right\}.$$

Exercise 1.5.6. *Show that $\sigma e_T = e_{\sigma T}$.*

Define the Specht module $S^\lambda \subseteq M^\lambda$ by $S^\lambda = \sum_T e_T$ (note that this sum is not direct!). The previous exercise shows that S^λ is, in fact, a module. Since S_n acts transitively on λ-tableaux, the previous exercise also shows

Theorem 1.5.7. *The Specht module S^λ is cyclically generated by any e_T: $S^\lambda = FS_n e_T$.*

Lemma 1.5.8. *Assume that $ld, \mu \in P(n)$, T_1 is a λ-tableaux, T_2 is a μ-tableaux, and $\kappa_{T_1}\{T_2\} \neq 0$. Then $\lambda \trianglerighteq \mu$.*

Proof: Lemma 1.4.1 says the following: If whenever $(ij) \in R(T_2)$, $(ij) \notin C(T_1)$, then $\lambda \trianglerighteq \mu$.

Assume that $(ij) \in R(T_2)$. Then,

$$(1-(ij))\{T_2\} = 0.$$

We'll show $(ij) \notin C(T_1)$. Indeed, assume $(ij) \in C(T_1)$. Let $\sigma_1, \ldots, \sigma_k$ be coset representative for $C(T_1)/\langle (ij) \rangle$. Then,

$$\kappa_{T_1} = \sum \operatorname{sgn}(\sigma_i)\sigma_i(1-(ij))$$

so $\kappa_{T_1}\{T_2\} = 0$. ∎

Lemma 1.5.9. *If T_1 and T_2 are λ-tableaux and $\kappa_{T_1}\{T_2\} \neq 0$, then $\kappa_{T_1}\{T_2\} = \pm e_{T_1}$.*

Proof: By construction, $\{T_2\}$ appears as a summand of e_{T_1}. The fact that $\kappa_{T_1}\{T_2\} \neq 0$ implies that $\{T_2\} = \{\sigma T_1\}$ for some $\sigma \in C(T_1)$. Hence, using the next exercise we deduce that

$$\kappa_{T_1}\{T_2\} = \kappa_{T_1}\{\sigma T_1\} = \kappa_{T_1}\sigma\{T_1\} = \pm e_{T_1}.$$

∎

Exercise 1.5.10. *Show that $\sigma\kappa_T = \kappa_T\sigma = \operatorname{sgn}(\sigma)\kappa_T$ for all $\sigma \in C(T)$.*

Corollary 1.5.11. *If $u \in M^\lambda$, the $\kappa_T u$ is a scalar multiple of e_T.*

Proof: u is a linear combination of λ-tabloids, so the result follows from the previous lemma. ∎

Define a bilinear form $\langle \cdot, \cdot \rangle : M^\lambda \otimes M^\lambda \to F$ by

$$\langle \{T_1\}, \{T_2\} \rangle = \begin{cases} 1 & \text{if } \{T_1\} = \{T_2\} \\ 0 & \text{else.} \end{cases}$$

This form is symmetric, non-degenerate and FS_n-invariant!

Exercise 1.5.12. *Assume the $x = \sum_{\sigma \in S_n} a_\sigma \sigma \in FS_n$ satisfies $a_\sigma = a_{\sigma^{-1}}$ for all $\sigma \in S_n$. (for example, ρ_T and κ_T are such elements). Then, for all $u, v \in M^\lambda$,*

$$\langle x.u, v\rangle = \langle u, x.v\rangle.$$

Theorem 1.5.13. *(Submodule Theorem) If $U \subseteq M^\lambda$ is a submodule, then $U \supseteq S^\lambda$ or $U \subseteq (S^\lambda)^\perp$.*

Proof: Assume $u \in U$. By the previous Corollary, $\kappa_T u$ is a scalar multiple of e_T. If T can be chosen so that this scalar is nonzero, then $U \supseteq S^\lambda$ because S^λ is cyclically generated by e_T. Otherwise, using the previous exercise, it follows that for all λ-tableaux T,

$$0 = \langle \kappa_T u, \{T\}\rangle = \langle u, \kappa_T \{T\}\rangle = \langle u, e_T\rangle.$$

Hence, $u \in (S^\lambda)^\perp$. ∎

Theorem 1.5.14. *Let $D^\lambda = S^\lambda / S^\lambda \cap (S^\lambda)^\perp$. Then, D^λ is either 0 or absolutely irreducible. If D^λ is nonzero, then $S^\lambda \cap (S^\lambda)^\perp$ is the unique maximal submodule of S^λ and D^λ is self-dual.*

Proof: By the Submodule Theorem, any submodule of S^λ is either S^λ itself, or contained in $S^\lambda \cap (S^\lambda)^\perp$. Therefore, if D^λ is nonzero, $S^\lambda \cap (S^\lambda)^\perp$ is the unique maximal submodule of S^λ and D^λ is irreducible.

The fact that D^λ is self-dual follows from general nonsense about FS_n-invariant bilinear forms.

It remains to prove that D^λ is absolutely irreducible. This means that if K/F is a field extension, then $K \otimes_F D^\lambda$ is irreducible.

Now, we have two irreducible modules $S_F^\lambda / S_F^\lambda \cap (S_F^\lambda)^\perp$ and $S_K^\lambda / S_K^\lambda \cap (S_K^\lambda)^\perp$. To prove that D^λ is absolutely irreducible, it is enough to show that

$$\dim_F S_F^\lambda / S_F^\lambda \cap (S_F^\lambda)^\perp = \dim_K S_K^\lambda / S_K^\lambda \cap (S_K^\lambda)^\perp.$$

Let $e_{T_1}, \dots, e_{T_N}$ be a basis for S_F^λ (also for S_K^λ). Both dimensions above equal the rank of the matrix $A = (\langle e_{T_i}, e_{T_j}\rangle)_{ij}$. But, the entries of this matrix lie in the prime subfield (because the e_{T_i} are linear combinations of tabloids with coefficients ± 1, and tabloids form an orthonormal basis for this form). Hence, the dimensions are equal. ∎

1.6. **Irreducible Representations of $\mathbb{Q}S_n$.** I'm not really going to write this section at this point. Let me just say that the S_n-invariant bilinear form on $M_\mathbb{Q}^\lambda$ is actually an inner product. As a consequence, $S^\lambda \cap (S^\lambda)^\perp = 0$, and therefore, S^λ is irreducible for each partition λ. It is also easy to prove that $S^\lambda \not\cong S^\mu$ if $\lambda \neq \mu$. We will prove this fact more generally in the context of abitrary fields.

It follows from these remarks that the set $\{S^\lambda | \lambda \in P(n)\}$ is a complete set of pairwise nonisomorphic irredudible representations of $\mathbb{Q}S_n$. Since all these modules are absolutely irreducible, it follows that $\mathbb{Q}$ is a splitting field for S_n.

1.7. **A result over $\mathbb{Q}$.** In the next few sections, we will see that many of the results for FS_n can be obtained from results about $\mathbb{Q}S_n$. Indeed, consider a '$\mathbb{Z}$-form': $M_\mathbb{Z}^\lambda = \bigoplus \mathbb{Z}\{T\} \subset M_\mathbb{Q}^\lambda$. Obviously, $M_F^\lambda \cong M_\mathbb{Z}^\lambda \otimes_\mathbb{Z} F$. What is more, the polytabloids e_T are linear combinations of tabloids with coefficients ± 1. In particular, we'll see that there is a natural $\mathbb{Z}$-form $S_\mathbb{Z}^\lambda$ for Specht modules, and $S_F^\lambda \cong S_\mathbb{Z}^\lambda \otimes_\mathbb{Z} F$.

In this section, we prove a result for Specht modules over $\mathbb{Q}$ that is interesting in its own right.

In the following sections, we construct the $\mathbb{Z}$-form for Specht modules promised above and, as an application, deduce an analogous result for Specht modules over an arbitrary field.

To begin, given a partition λ, let λ^t be the transpose partition:

$$\lambda = \begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \end{array}, \quad \lambda^t = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \\ \cline{1-1} \end{array}.$$

Given a λ tableaux T, let T^t denote the corresponding λ^t-tableaux:

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}, \quad T^t = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1} \end{array}.$$

Theorem 1.7.1. $(S_{\mathbb{Q}}^{\lambda^t})^* \cong S_{\mathbb{Q}}^{\lambda} \otimes S^{(1^n)}$.

Remark 1.7.2. *In characteristic 0, Specht modules are self-dual, however,* $S_{\mathbb{Z}}^{\lambda} \ncong (S_{\mathbb{Z}}^{\lambda})^*$.

Proof: Let $u \in S^{(1^n)}$ be a nonzero element of the 1-dimensional sign representation. Fix a λ-tableaux T and recall that

$$M^\lambda = \mathbb{Q}S_n.\{T\}.$$

Therefore, we may define a map

$$\theta : M^{\lambda^t} \to M^\lambda \otimes S^{(1^n)}$$

by $\theta(\{T^t\}) = \rho_{T^t}(\{T\} \otimes u)$. For an arbitrary λ^t-tabloid $\{S^t\}$, write $S^t = \sigma T^t$ for some $\sigma \in \mathbb{Q}S_n$ and define

$$\theta(\{S^t\}) = \sigma\theta(\{T^t\}).$$

Note that

$$\rho_{T^t} = \sum_{w \in R(T^t)} w = \sum_{w \in C(T)} w.$$

In particular, if $\{S^t\} = \{T^t\}$, then $S^t = \sigma T^t$ for some $\sigma \in R(T^t)$. Hence,

$$\theta(\{S^t\}) = \sigma\rho_{T^t}(\{T\} \otimes u) = \theta(\{T^t\})$$

since $\sigma\rho_{T^t} = \rho_{T^t}$. That is, θ is well-defined.

Note also,

$$\begin{aligned} \rho_{T^t} w(\{T\} \otimes u) &= \sum_{w \in C(T)} (\{wT\} \otimes \operatorname{sgn}(w)w) \\ &= \left(\sum_{w \in C(T)} \operatorname{sgn}(w) w\{T\} \right) \otimes u \\ &= e_T \otimes u. \end{aligned}$$

Hence, for an arbitrary λ^t-tableaux $S^t = \sigma T^t$,

$$\theta(\{S^t\}) = \operatorname{sgn}(\sigma) e_S \otimes u.$$

We deduce that $\operatorname{im}\theta = S^\lambda \otimes S^{(1^n)}$.

Next, we want to show that $\mathrm{im}\theta|_{S^{\lambda^t}} \neq 0$. It follows that $\theta|_{S^{\lambda^t}}$ is injective, since S^{λ^t} is irreducible. Indeed, $S^{\lambda^t} = \mathbb{Q}S_n.e_T$, so it is enough to show that $\theta(e_{T^t}) \neq 0$. We have

$$\begin{aligned}\theta(e_{T^t}) &= \kappa_{T^t}\theta(\{T^t\}\\ &= \kappa_{T^t}\rho_{T^t}(\{T\}\otimes u)\\ &= (\rho_T\kappa_T\{T\})\otimes u.\end{aligned}$$

Hence, we are left to show that $\rho_T\kappa_T\{T\} \neq 0$. Indeed,

$$\langle \rho_T\kappa_T\{T\}, \{T\}\rangle = \langle e_T, \rho_T\{T\}\langle = |R(T)|\langle e_T, \{T\}\rangle = |R(T)| \neq 0.$$

Finally, to conclude that $\theta|_{S^{\lambda^t}}$ is an isomorphism, we compute that $\dim S^{\lambda^t} = \dim S^\lambda$. We have

$$\dim S^\lambda \leq \dim \mathrm{im}\theta = \dim S^\lambda \otimes S^{(1^n)} = \dim S^\lambda.$$

Interchanging the roles of λ and λ^t in the construction above shows that $\dim S^{\lambda^t} \leq \dim S^\lambda$, so there is equality. ∎

1.8. **Standard Bases of Specht Modules.** We again work over an arbitrary field F. As usual, fix a partition λ. The Specht module S^λ is spanned by the set of λ-polytabloids $\{e_T\}$, however, this set is not linearly independent. We now begin the work of picking out a basis for S^λ from this set.

To this end, define a total ordering on tabloids $\{T_1\} < \{T_2\}$ if for some $i = 1, \ldots, n$,

(1) when $j > i$, j is in the same row of $\{T_1\}$ and $\{T_2\}$;
(2) the number i appears in a higher row of $\{T_1\}$ than $\{T_2\}$.

A nice way to think about this ordering is as follows: given a tabloid $\{T_1\}$, one can make a larger tabloid $\{T_2\}$ by exchanging a smaller number in a lower row for a larger number in a higher row:

$$\left\{\begin{array}{|c|c|c|}\hline 3&4&5\\\hline 1&2\\\cline{1-2}\end{array}\right\} < \left\{\begin{array}{|c|c|c|}\hline 2&4&5\\\hline 1&3\\\cline{1-2}\end{array}\right\} < \left\{\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&3\\\cline{1-2}\end{array}\right\} < \left\{\begin{array}{|c|c|c|}\hline 2&3&5\\\hline 1&4\\\cline{1-2}\end{array}\right\} < \cdots.$$

Note that third inequality shows that the heuristic above is not equivalent to the definition.

Now, call a tableaux *standard* if its entries increase along rows and down columns. Then, a *standard tabloid* is one which contains a standard tableaux, and a *standard polytabloid* is one which contains a standard tabloid.

Notice that each standard tabloid contains a unique standard tableaux, but a standard polytabloid may contain multiple standard tabloids. For example,

$$T = \left\{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\right\}$$

$$e_T = \left\{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&5\\\cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\right\} + \left\{\begin{array}{|c|c|c|}\hline 3&4&5\\\hline 1&2\\\cline{1-2}\end{array}\right\}.$$

Exercise 1.8.1. *If T is standard, then $\{T\}$ is the largest tabloid in e_T. Deduce that the set $\{e_T | T$ standard$\}$ is linearly independent.*

Hint: Let $\{T_1\} < \ldots < \{T_N\}$ be a complete list of standard tabloids, where $T_1, \ldots, T_N$ are standard. Then e_{T_k} cannot be written as a linear combination of e_{T_j} for $j < k$.

Given this exercise, the remainder of this section is devoted to showing the following:

Theorem 1.8.2. *The set* $\{e_T | T \text{ standard}\}$ *is a basis for* S^λ.

Proof: Let $[T] = C(T)T$ be the column equivalence class of T. There is an analogous ordering on the set of column equivalence classes as on the set of tabloids. Roughly, $[T_1] < [T_2]$ if T_1 is obtained from T_2 by bumping smaller entries left.

Now, Suppose that T is not a standard tableaux. We will assume that $e_{T'}$ can be written as a product of standard polytabloids whenever $[T'] < [T]$, and then prove the result for e_T. To begin, since $\pi e_T = \pm e_T$ for all $\pi \in C(T)$, we may assume that the entries of T increase down columns.

Now, if T is not standard, then there are a pair of columns in T of the form

$$\begin{array}{|c|c|}
\hline a_1 & b_1 \\ \hline
\vdots & \vdots \\ \hline
a_q & b_q \\ \hline
\vdots & \vdots \\ \hline
a_s & b_s \\ \hline
\vdots & \\ \cline{1-1}
a_r & \\ \cline{1-1}
\end{array}$$

where $a_r > \cdots > a_q > b_q > \cdots > b_s$.

To prove the result, let $X = \{a_q, \ldots, a_r\}$ and $Y = \{b_1, \ldots, b_q\}$ and introduce the associated *Garnir element*

$$G_{X,Y} = \sum_{i=1}^{k} \operatorname{sgn}(\sigma_i)\sigma_i$$

where the sum is over coset representatives $\sigma_1, \ldots, \sigma_k$ for $S_{X\cup Y}/S_X \times S_Y$. Observe that these coset representatives may be chosen so that the entries of $\sigma_i T$ are increasing down columns for all $i = 1, \ldots, k$. In particular, we may assume $\sigma_1 = 1$.

The Garnir element is designed so that $G_{X,Y}e_T = 0$. We will sketch a proof of this following this proof. Assuming this fact, we have

$$0 = G_{X,Y}e_T = \sum_{i=1}^{k} \operatorname{sgn}(\sigma_i)e_{\sigma_i T} = e_T + \sum_{i=2}^{k} \operatorname{sgn}(\sigma_i)e_{\sigma_i T}. \tag{2}$$

Since $b_1 < \cdots < b_q < a_q < \cdots < a_r$, $[\sigma_i T] < [T]$ for all $i > 1$. Thus, (2) and the inductive hypothesis together imply

$$e_T = \sum_{[T']<[T]} a_{T'}e_{T'}.$$

Moreover, by considering the form of the Garnir element, we deduce that the $a_{T'}$ belong to the prime subfield of F! ∎

Remark 1.8.3. *This remark is devoted to prove that* $G_{X,Y}e_T = 0$ *in the proof above.*

To this end, fix an arbitrary partition μ*, and let* S *be a* μ *tableaux. Let* X *be a subset of the ith column of* μ *and* Y *a subset of the* $(i+1)$*th column of* μ*. Note that the height of column* i *(resp.* $i+1$*) is* μ_i^t *(resp.* μ_{i+1}^t*). Let* $G_{X,Y}$ *be the Garnir element defined as in the proof above.*

Assume that $|X \cup Y| \geq \mu_i^t$, *so that* X *and* Y *must intersect a common row of* S. *It follows that for every* $\tau \in C(S)$, *there is a pair of numbers in* $X \cup Y$ *which are in the same row of* τS. *Let*

$$\Sigma(X \cup Y) = \sum_{w \in S_{X \cup Y}} \operatorname{sgn}(w) w.$$

Then,

$$\Sigma(X \cup Y)\{\tau S\} = 0 \textit{ (Prove it!).}$$

Taking the alternating sum over all elements of $C(S)$ *yields*

$$0 = \Sigma(X \cup Y)\kappa_S\{S\} = \Sigma(X \cup Y)e_S.$$

Now, let

$$\Sigma(X \times Y) = \left(\sum_{\sigma \in S_X \times S_Y} \operatorname{sgn}(\sigma)\sigma \right)$$

and let $\sigma_1, \ldots, \sigma_N$ *be coset representative for* $S_X \times S_Y \backslash S_n$. *Then*

$$\kappa_S = \Sigma(X \times Y) \left(\sum_{i=1}^{N} \operatorname{sgn}(\sigma_i)\sigma_i \right)$$

while

$$\Sigma(X \cup Y) = G_{X,Y}\Sigma(X \times Y).$$

Hence,

$$\begin{aligned}
0 &= \Sigma(X \cup Y)e_S \\
&= G_{X,Y}\Sigma(X \times Y)^2 \left(\sum_{i=1}^{N} \operatorname{sgn}(\sigma_i)\sigma_i \right) \{S\} \\
&= |X|!|Y|!G_{X,Y}\Sigma(X \times Y) \left(\sum_{i=1}^{N} \operatorname{sgn}(\sigma_i)\sigma_i \right) \{S\} \\
&= |X|!|Y|!G_{X,Y}\kappa_S\{S\} \\
&= |X|!|Y|!G_{X,Y}e_S.
\end{aligned}$$

The previous theorem has a number of remarkable corollaries.

Corollary 1.8.4. $\dim_F S_F^\lambda$ *is independent of* F *and equals the number of standard* λ*-tableaux.*

Corollary 1.8.5. *In* $S_{\mathbb{Q}}^\lambda$, *each polytabloid can be written as an* integral *linear combination of standard polytabloids.*

Corollary 1.8.6. *The entries in the matrices representing* S_n *over* $\mathbb{Q}$ *with respect to the standard basis of* $S_{\mathbb{Q}}^\lambda$ *have integer entries.*

Corollary 1.8.7. *If* $v \in S^\lambda$ *is nonzero, then the largest tabloid occurring in* v *is standard.*

Corollary 1.8.8. *If* $v \in S^\lambda$ *and the tabloids involved in* v *occur with integer coefficients, then* v *is an integral linear combination of standard polytabloids.*

Proof: Say $v = \sum a_{\{T\}}\{T\}$. By Corollary 1.8.7, the largest tabloid, $\{T_0\}$ occuring in v is standard. Now,

$$v - a_{\{T_0\}}e_{T_0} = \sum_{\{T\}<\{T_0\}} b_{\{T\}}\{T\}.$$

Indeed, if S has numbers increasing down columns, then all the tabloids $\{S'\}$ occurring in e_S satisfy $\{S'\} < \{S\}$. ∎

Corollary 1.8.9. *If $v \in S^\lambda_{\mathbb{Q}}$ is an integral combination of tabloids then we can reduce coefficients mod p to obtain an element of $S^\lambda_{F_p}$, where F_p is the field with p elements. That is, $S^\lambda_{F_p}$ is the p-modular representation obtained from $S^\lambda_{\mathbb{Q}}$.*

1.9. **Modular Representations.** The results of the previous section are somewhat striking. In order to study the representation theory of the symmetric group over a field of characteristic p, one should construct representations over $\mathbb{Z}$ and then reduce scalars mod p. As an application of this idea, we prove

Theorem 1.9.1. *Over any field: $S^\lambda \otimes S^{(1^n)} \cong (S^{\lambda^t})^*$.*

To prove this, we need the following technical lemma:

Lemma 1.9.2. *Suppose $\theta \in \mathrm{Hom}_{\mathbb{Q}S_n}(M^\lambda_{\mathbb{Q}}, M^\mu_{\mathbb{Q}})$ and for all $\{T\} \in M^\lambda_{\mathbb{Q}}$, $\theta(\{T\})$ is an integral linear combination of μ-tabloids. Then, reducing mod p yields an element $\overline{\theta} \in \mathrm{Hom}_{F_p}(M^\lambda_{F_p}, M^\mu_{F_p})$. Moreover, if $\ker\theta = (S^\lambda_{\mathbb{Q}})^\perp$, then $\ker\overline{\theta} \supset (S^\lambda_{F_p})^\perp$.*

Proof: Obviously, $\overline{\theta} \in \mathrm{Hom}_{F_p}(M^\lambda_{F_p}, M^\mu_{F_p})$. Now, choose a basis $f_1, \ldots, f_k$ for $(S^\lambda_{\mathbb{Q}})^\perp$ and extend it by the standard basis of $S^\lambda_{\mathbb{Q}}$ to a basis $f_1, \ldots, f_m$ for $M^\lambda_{\mathbb{Q}}$. Let $\{T_1\}, \ldots, \{T_m\}$ be the basis for $M^\lambda_{\mathbb{Q}}$ given by λ-tabloids.

Define the matrix $N = (n_{ij})$ by $n_{ij} = \langle f_i, \{T_j\}\rangle$. In other words, the ith row of N gives the expansion of f_i in terms of the $\{T_j\}$. By row reducing the first k rows and rescaling we may assume that all $n_{ij} \in \mathbb{Z}$ and the first k rows are linearly independent modulo p.

Now, reduce all entries mod p, we get a set of vectors in $M^\lambda_{F_p}$. The last $m-k$ vectors are the standard basis of $S^\lambda_{F_p}$, while the first k are linearly independent and orthogonal to the last $m-k$ vectors. Hence, we have constructed a basis for $(S^\lambda_{F_p})^\perp$ by reduction modulo p, which implies $\overline{\theta}((S^\lambda_{F_p})^\perp) = 0$. ∎

Now, it is enough to prove the theorem for $F = F_p$. Observe that we have already constructed a homomorphism $\theta : M^{\lambda^t}_{\mathbb{Q}} \to M^\lambda_{\mathbb{Q}} \otimes S^{(1^n)}_{\mathbb{Q}}$. We have $\mathrm{im}\theta = S^\lambda_{\mathbb{Q}} \otimes S^{(1^n)}_{\mathbb{Q}}$ and $\ker\theta = (S^\lambda_{\mathbb{Q}})^\perp$.

Applying the previous lemma, we obtain a homomorphism $\overline{\theta}$. Recall that we deduced that $\theta|_{S^{\lambda^t}_{\mathbb{Q}}}$ was bijective by comparing dimensions. It now follows by Corollary 1.8.4 that $\ker\overline{\theta} = (S^{\lambda^t}_{F_p})^\perp$.

1.10. **Irreducible Representations of FS_n.** In this section, we classify all irreducible representations of FS_n, where F is an arbitrary field of characteristic $p > 0$. We begin with the following definition:

Definition 1.10.1. *Let G be a finite group. Call a conjugacy class p-regular class if the order of its elements are coprime to p.*

To facilitate the classification, we use the following facts from Curtis/Reiner:

Theorem 1.10.2. *(1) (Curtis/Reiner 82.6) The number of absolutely irreducible representations of a finite group G equals the number of p-regular classes.*

(2) (Curtis/Reiner 83.7) If $\mathbb{Q}$ is a splitting field for G, then every field is a splitting field for G.

Now, a conjugacy class in S_n is labelled by a partition λ, and the order of the elements in the λ-conjugacy class is $\operatorname{lcm}(\lambda_1, \lambda_2, \ldots)$. In particular, λ labels a p-regular class if, and only if, no part of λ is divisible by p.

Definition 1.10.3. *(1) Call a partition p-regular if no part is repeated p times (i.e., there is no i such that $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p}$). Let $P_p(n)$ denote the set of p-regular partitions.*

(2) Call a partition p-class regular if no part is divisible by p. Let $P_p^*(n)$ denote the set of p-class regular partitions.

Using the theorems above, we understand that the number of absolutely irreducible representations of FS_n ($\mathrm{Char}F = p$) equals $|P_p^*(n)|$. This, in turn is the number of irreducible representations, since $\mathbb{Q}$ is a splitting field for S_n. In the remainder of the section, we will prove that the module D_F^λ is nonzero precisely when $\lambda \in P_p(n)$ and $D_F^\lambda \not\cong D_F^\mu$ if $\lambda \neq \mu$. The following lemma shows that this gives a complete classification of irreducible FS_n-modules:

Lemma 1.10.4. $|P_p(n)| = |P_p^*(n)|$.

Proof: Define the Glashier map $G : P_p(n) \to P_p^*(n)$ as follows: given $\lambda = (\lambda_1, \ldots, \lambda_N) \in P_p(n)$, $G(\lambda)$ has parts $g(\lambda_1), \ldots, g(\lambda_N)$, where

$$g(\lambda_i) = \underbrace{q, \ldots, q}_{p^k}$$

if $\lambda_i = p^k q$ with $(q, p) = 1$.

To write down the inverse, $G^{-1} : P_p^*(n) \to P_p(n)$, given $\lambda \in P_p^*(n)$, write $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \ldots)$, where $m_i(\lambda)$ is the number of parts of λ which equal i. For each i, let

$$m_i(\lambda) = m_i^{(0)}(\lambda) + p m_i^{(1)}(\lambda) + p^2 m_i^{(2)}(\lambda) + \cdots$$

where $0 \leq m_i^{(j)}(\lambda) < p$. This is the p-adic expansion of $m_i(\lambda)$. Then,

$$G^{-1}(\lambda) = \sum_{j \geq 0} ((p^j \cdot 1)^{m_1^{(j)}(\lambda)} (p^j \cdot 2)^{m_2^{(j)}(\lambda)} \ldots).$$

where the sum has a more (or less) obvious meaning. ∎

Exercise 1.10.5. *Check that G and G^{-1} are actually inverse maps.*

We now set out to determine when $D_F^\lambda = S_F^\lambda / S_F^\lambda \cap (S_F^\lambda)^\perp \neq 0$. To this end, we appeal to the fact that S_F^λ is obtained from $S_\mathbb{Z}^\lambda$ (i.e. $\mathbb{Z}$-linear combinations of standard polytabloids) by reduction modulo p. Define the matrix

$$A_\mathbb{Z} = (\langle e_T, e_{T'} \rangle)_{T, T' \text{ standard } \lambda\text{-tableaux}},$$

where $e_T \in S_\mathbb{Z}^\lambda$. Define $A_F = A_\mathbb{Z} \otimes_\mathbb{Z} F$.

Recall that

$$\dim_F D_F^\lambda = \dim_F S_F^\lambda / S_F^\lambda \cap (S_F^\lambda)^\perp = \mathrm{rank} A_F.$$

Therefore, $D_F^\lambda = 0$ if, and only if $A_F = 0$.

Let $g^\lambda = \gcd\{\langle e_T, e_{T'}\rangle | T, T' \text{ standard } \lambda\text{-tableaux}\}$. That is, g^λ is the greatest common divisor of the entries in $A_{\mathbb{Z}}$. Then,

$$D_F^\lambda = 0 \text{ if, and only if, } p | g^\lambda.$$

Lemma 1.10.6. $\prod_{j\geq 1} m_j(\lambda)!$ *divides* g^λ, *and* g^λ *divides* $\prod_{j\geq 1}(m_j(\lambda)!)^j$

Proof: Define an equivalence relation on the set of λ-tabloids: $\{T_1\} \sim \{T_2\}$ if, and only if, for all i, j, i and j belong to the same row of T_2 whenever they belong to the same row of T_1. That is, T_2 is obtained from T_1 by shuffling rows of the same length. Observe that the size of such an equivalence class is $\prod_j m_j(\lambda)!$.

Now, if $\{T_1\}$ occurs in e_T, and $\{T_1\} \sim \{T_2\}$, then $\{T_2\}$ occurs in e_T, since it is obtained from $\{T_1\}$ via an element of $C(T)$. Moreover, the sign of the coefficient (± 1) of $\{T_1\}$ is either the same as the coefficient of $\{T_2\}$ or opposite. This is obvious. The point is, this sign depends only on $\{T_1\}$ and $\{T_2\}$, and NOT on $\{T\}$. That is, if $\{T_1\}$ and $\{T_2\}$ occur in e_T and in $e_{T'}$, then there coefficients either have the same sign in both cases, or opposite signs in both cases.

Now, if e_T and $e_{T'}$ have $\{T_1\}$ in common, then they have $\{T_2\}$ in common for every $\{T_2\} \sim \{T_1\}$. Hence, the number of tabloids in common is a multiple of $\prod_j m_j(\lambda)!$. Moreover, as stated before, $\{T_1\}$ and $\{T_2\}$ occur with either the same or opposite signs in both e_T and $e_{T'}$. It follows that

$$\langle e_T, e_{T'}\rangle = \text{ a multiple of } \prod_{j\geq 1} m_j(\lambda)!.$$

This prove $\prod_{j\geq 1} m_j(\lambda)! | g^\lambda$.

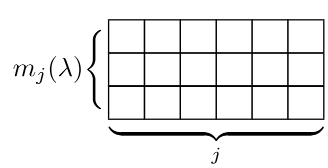
We now prove $g^\lambda | \prod_{j\geq 1}(m_j(\lambda)!)^j$. To this end, given a tableaux T, let T^* be the tableaux obtained by reversing the entries along rows:

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \cline{1-3} 8 & 9 & 10 & \\ \cline{1-3} 11 & & & \\ \cline{1-1}\end{array} \quad \text{and} \quad T^* = \begin{array}{|c|c|c|c|}\hline 4 & 3 & 2 & 1 \\ \hline 7 & 6 & 5 & \\ \cline{1-3} 10 & 9 & 8 & \\ \cline{1-3} 11 & & & \\ \cline{1-1}\end{array} .$$

What tabloids to e_T and e_{T*} have in common?

The answer is that $\{\sigma T\}$ occurs in e_T and e_{T^*} if an only if σ permutes entries between rows of the same length. For example, in the tableaux above, we see that all tabloids in e_{T*} must have 1 in the first row. Hence, no common tabliod has 5,8 or 11 in the first row. We conclude that every common tabloid must have 2 in row 1, so no common tabloid has 6 or 9 in row 1. Hence, every common tabloid must have 3 in row 1, and so on...

We now conclude that $\langle e_T, e_{T*}\rangle = \prod_j (m_j(\lambda)!)^j$. Indeed, the number of tabloids in common can be computed as follows: for each block



one obtains a common tabloid by acting on each column with a copy of $S_{m_j(\lambda)}$. There are j columns, which give a contribution of $(m_j(\lambda)!)^j$ common tabloids for each such block... ∎

Corollary 1.10.7. *$(p, g^\lambda) = 1$ if, and only if, $\lambda \in P_p(\lambda)$.*

Corollary 1.10.8. $\kappa_T e_{T*} = \left(\prod_j (m_j(\lambda)!)^j\right) e_T$.

Proof: We have already proved that $\kappa_T\{T'\} = \varepsilon e_T$, $\varepsilon \in \{-1, 0, 1\}$, for T, T' λ-tableaux. Hence,

$$\kappa_T e_{T*} = h e_T$$

for some $h \in \mathbb{Z}$. Now, $\{T^*\} = \{T\}$, so

$$h = h\langle e_{T^*}, \{T\}\rangle = \langle \kappa_T e_{T^*}, \{T\}\rangle = \langle e_{T^*}, \kappa_T\{T\}\rangle = \langle e_{T^*}, e_T\rangle.$$

Hence, the result. ∎

The following theorem is just a restatement of Corollary 1.10.7.

Theorem 1.10.9. *$D_F^\lambda \neq 0$ if, and only if $\lambda \in P_p(n)$.*

We now have the right number of irreducible FS_n-modules. We will be done as soon as we show that they are all pairwise non-isomorphic.

Lemma 1.10.10. *Assume $\lambda \in P_p(n)$, $\mu \in P(n)$, $U \subseteq M^\mu$ is a submodule, and $\theta \in \mathrm{Hom}_{FS_n}(S^\lambda, M^\mu/U)$ is nonzero. Then $\lambda \trianglerighteq \mu$ and if $\lambda = \mu$, $\mathrm{im}\theta \subseteq S^\mu + U/U$.*

Proof: Let T be a λ-tableaux and T^* as before. Then, $\kappa_T e_{T^*} = h e_T$, where $h = \prod_j (m_j(\lambda)!)^j \neq 0 \in F$ because λ is p-regular. Also, $\theta(e_T) \neq 0$ because e_T cyclically generates S^λ and $\theta \neq 0$. Hence,

$$0 \neq h\theta(e_T) = \theta(h e_T) = \theta(\kappa_T e_{T^*}) = \kappa_T \theta(e_{T^*}).$$

But, $\theta(e_{T*})$ is a linear combination of $\{S\} + U$, where $\{S\}$ is a μ-tabloid, and $\kappa_T\{S\} \neq 0$ implies that $\lambda \trianglerighteq \mu$ by Lemma 1.5.8.

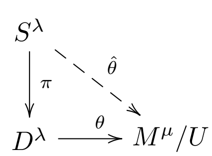
Moreover, if $\lambda = \mu$, then

$$\theta(e_T) = h^{-1}\kappa_T\theta(e_{T^*}).$$

As before, we observe that $\theta(e_{T^*})$ is a linear combination of $\{S\} + U$, where $\{S\}$ is a λ-tabloid. It follows that $\kappa_T\theta(e_{T^*}) = a e_T + U \in S^\lambda + U/U$ for some $a \in F$ by Lemma 1.5.9. Finally, the result follows because S^λ is cyclic. ∎

Corollary 1.10.11. *Let $\lambda \in P_p(n)$, $\mu \in P(n)$. Assume $U \subseteq M^\mu$ is a submodule and $\theta \in \mathrm{Hom}_{FS_n}(D^\lambda, M^\mu/U)$ is nonzero. Then $\lambda \trianglerighteq \mu$, and $\lambda \triangleright \mu$ if $U \supset S^\mu$.*

Proof: We have



where $\hat{\theta} = \theta \circ \pi \in \mathrm{Hom}_{FS_n}(S^\lambda, M^\mu/U)$ is nonzero. This implies that $\lambda \trianglerighteq \mu$. Moreover, if $\lambda = \mu$, then $\hat{\theta}$ nonzero implies that $S^\lambda + U/U \neq 0$ (i.e. $U \not\supseteq S^\lambda$). ∎

Theorem 1.10.12. *The set $\{D^\lambda | \lambda \in P_p(n)\}$ forms a complete set of pairwise non-isomorphic simple FS_n-modules.*

Proof: Assume $D^\lambda \cong D^\mu$. Then we get a nonzero $\theta \in \mathrm{Hom}_{FS_n}(D^\lambda, M^\mu/S^\mu \cap (S^\mu)^\perp)$ via the composition

$$D^\lambda \xrightarrow{\cong} D^\mu \hookrightarrow M^\mu/S^\mu \cap (S^\mu)^\perp \ .$$

Hence, $\lambda \trianglerighteq \mu$. Interchanging the roles of λ and μ implies that $\mu \trianglerighteq \lambda$. Hence, $\lambda = \mu$. ∎

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