Assignment 5: Selected Solutions

Section 3.4

17. Let $\varphi : G_1 \to G_2$ be a group isomorphism. Prove that if H is a subgroup of G_1, then $\varphi(H)$ is a subgroup of G_2.

Proof: Let $e_i \in G_i$ $(i = 1, 2)$ be the identity element. Since H is a subgroup, $e_1 \in H$. Hence, $e_2 = \varphi(e_1) \in \varphi(H)$. Next, if $x, y \in \varphi(H)$, then $x = \varphi(a)$ and $y = \varphi(b)$ for some $a, b \in H$ by definition of $\varphi(H)$. It follows that $xy^{-1} = \varphi(a)\varphi(b)^{-1} = \varphi(ab^{-1}) \in \varphi(H)$, showing that $\varphi(H)$ is a subgroup of G_2. ∎

26. Show that the map $\varphi : GL_2(\mathbb{R}) \to \mathbb{R}^\times$ is a group homomorphism

Proof: Let $A, B \in GL_2(\mathbb{R})$. Then, $\varphi(AB) = \det(AB) = \det(A)\det(B) = \varphi(A)\varphi(B)$. Hence, φ is a homomorphism. ∎

Section 3.7

8. Let $\varphi : G_1 \to G_2$ and $\theta : G_2 \to G_3$ be group homomorphisms. Show that $\theta\varphi : G_1 \to G_3$ is a group homomorphism, and $\ker\varphi \subseteq \ker\theta\varphi$.

Proof: Let $a, b \in G_1$. Then,

$$\theta\varphi(ab) = \theta(\varphi(ab)) = \theta(\varphi(a)\varphi(b)) = \theta(\varphi(a))\theta(\varphi(b)) = \theta\varphi(a)\theta\varphi(b)$$

showing that $\theta\varphi$ is a group homomorphism. Now, for $i = 1, 2, 3$, let $e_i \in G_i$ denote the identity element. If $a \in \ker\varphi$, then $\varphi(a) = e_2 \in G_2$. Hence,

$$\theta\varphi(a) = \theta(\varphi(a)) = \theta(e_2) = e_3$$

and $a \in \ker\theta\varphi$. ∎

16. Let G be a finite group with $|G| = 2n$. Show that if H is a subgroup of G with n elements, then H is normal in G.

Proof: Let $\varphi : G \to \{1, -1\}$ be the map $\varphi(x) = 1$ if $x \in H$ and $\varphi(x) = -1$ if $x \in G - H$. Here, we regard $\{1, -1\}$ as a group under multiplication.

We show that φ is a group homomorphism. Certainly, we have

$$\varphi(xy) = \begin{cases} 1 & \text{if } x, y \in H; \\ -1 & \text{if } x \in H, y \in G - H \\ -1 & \text{if } x \in G - H, y \in H \end{cases}$$

so, $\varphi(xy) = \varphi(x)\varphi(y)$ in all these cases. Finally, assume that $x, y \in G - H$. Since $[G : H] = 2$, it follows that $x^{-1}H = yH$, so $xy \in H$. Thus, $\varphi(xy) = 1 = (-1)(-1) = \varphi(x)\varphi(y)$ and φ is a homomorphism.

It is left to observe that $H = \ker\varphi$ and apply the fundamental homomorphism theorem. ∎

18. Let D_n be given by generators r, j with $r^n = e$, $j^2 = e$ and $jr = r^{-1}j$. Show that any subgroup of $\langle r \rangle$ is normal in D_n.

Proof: Let $\langle r^k \rangle$ be any subgroup of $\langle r \rangle$. If $x \in D_n$, then $x = r^i$ or $x = r^i j$ for some $0 \leq i < n$. In the first case, $r^i r^k r^{-i} = r^k$, and in the second case, $r^i j r^k j r^{-i} = r^i r^{-k} j^2 r^i = r^{-k}$. Since $\langle r^k \rangle = \langle r^{-k} \rangle$ it follows that $\langle r^k \rangle$ is normal in D_n. ∎

20. Let G_1 and G_2 be groups.

(a) For $i = 1, 2$, define $\pi_i : G_1 \times G_2 \to G_i$ by $\pi_i((g_1, g_2)) = g_i$. Show that π_1 and π_2 are homomorphisms.

(b) Let G be any group and $\varphi : G \to G_1 \times G_2$ be a function. Show that φ is a homomorphism if, and only if $\pi_1\varphi$ and $\pi_2\varphi$ are group homomorphisms.

Proof: (a) Let $(g_1, g_2), (h_1, h_2) \in G_1 \times G_2$. Then, for $i = 1, 2$,

$$\pi_i((g_1, g_2)(h_1, h_2)) = \pi_i((g_1h_1, g_2h_2)) = g_ih_i = \pi_i((g_1, g_2))\pi_i((h_1, h_2))$$

showing that π_i is a homomorphism.

(b) First, assume that φ is a homomorphism. Then, for $i = 1, 2$, $\pi_i\varphi$ is a composition of two homomorphisms, and, hence, a homomorphism.

On the otherhand, suppose that $\pi_i\varphi$ is a homomorphism for $i = 1, 2$. Observe that

$$\varphi(g) = (\pi_1\varphi(g), \pi_2\varphi(g)).$$

It follows that

$$\begin{aligned} \varphi(gh) &= (\pi_1\varphi(gh), \pi_2\varphi(gh)) \\ &= (\pi_1\varphi(g)\pi_1\varphi(h), \pi_2\varphi(g)\pi_2\varphi(h)) \\ &= (\pi_1\varphi(g), \pi_2\varphi(g))(\pi_1\varphi(h), \pi_2\varphi(h)) \\ &= \varphi(g)\varphi(h) \end{aligned}$$

so φ is a homomorphism. ∎

Section 3.8

5. Let G be a group and H a subgroup of G. Prove that there is a bijection between the set of left and right cosets of H.

Proof: Define a function $f : G/H \to H\backslash G$ by $f(gH) = Hg^{-1}$.

This map is well defined. Indeed, assume $g_1H = g_2H$. This holds if, and only if, $g_1 = g_2h$ for some $h \in H$. But, inverting both sides of the equality yields $g_1^{-1} = h^{-1}g_2^{-1}$. This statement is equivalent to saying that $Hg_1^{-1} = Hg_2^{-1}$.

To see that this map is injective, note that all the statements in the argument above are equivalences (i.e. $g_1H = g_2H \Leftrightarrow Hg_1^{-1} = Hg_2^{-1}$).

Finally, this map is surjective, since given $Hg \in H\backslash G$, one has $f(g^{-1}H) = Hg$. ∎

23. Let G be the set of all matrices in $GL2(\mathbb{Z}_5)$ of the form $\begin{pmatrix} m & b \\ 0 & 1 \end{pmatrix}$.

(a) Show that G is a subgroup of $GL_2(\mathbb{Z}_5)$.

(b) Show that the subset N of all matrices of the form $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ is a normal subgroup of G.

(c) Show that G/N is cyclic of order 4.

Proof: (a) First, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G$. Moreover, if $x = \begin{pmatrix} m & b \\ 0 & 1 \end{pmatrix} \in G$, then

$$x^{-1} = \frac{1}{m}\begin{pmatrix} 1 & -b \\ 0 & m \end{pmatrix} = \begin{pmatrix} 1/m & -b/m \\ 0 & 1 \end{pmatrix} \in G.$$

Finally, Given $x = \begin{pmatrix} m_1 & b_1 \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} m_2 & b_2 \\ 0 & 1 \end{pmatrix}$ in G, one has

$$xy = \begin{pmatrix} m_1m_2 & m_1b_2 + b_1 \\ 0 & 1 \end{pmatrix} \in G.$$

Thus, G is a subgroup of $GL_2(\mathbb{Z}_5)$.

(b) To show that N is a subgroup, take $m, m_1, m_2 = 1$ in the above computations. To see that N is normal, observe that if $x = \begin{pmatrix} m & b \\ 0 & 1 \end{pmatrix} \in G$, and $y = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \in N$, then

$$\begin{aligned} xyx^{-1} &= \begin{pmatrix} m & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/m & -b/m \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & mc \\ 0 & 1 \end{pmatrix} \in N. \end{aligned}$$

For another proof that N is normal, see part (c) below.

(c) Define $\varphi : G \to \mathbb{Z}_5^\times$ by $\varphi\left(\begin{pmatrix} m & b \\ 0 & 1 \end{pmatrix}\right) = m$. Then, using the computation of xy in part (a), we deduce that φ is a homomorphism. Now, $\ker\varphi = \{x \in G | \varphi(x) = 1\} = N$ (which shows, in particular, that N is normal). By the fundamental homomorphism theorem, $G/N \equiv \mathbb{Z}_5^\times$, which is a cyclic group of order 4 (it is generated, for example by $[2]_5$–check!). ∎

27. Let H and N be subgroups of G, and assume that N is normal.

(a) N is a normal subgroup of HN.

(b) Each element of HN/N has the form hN for some $h \in H$.

(c) $\varphi : H \to HN/N$ defined by $\varphi(h) = hN$ is a surjective homomorphism.

(d) $HN/N \equiv H/H \cap N$.

Proof: (a) The fact that HN is a subgroup of G follows from the fact that N is normal. This was proved in section 3.3 under the weaker hypothesis that $hnh^{-1} \in N$ whenever $h \in H$ and $n \in N$. The fact that N is normal in HN is clear since it is normal in G.

(b) Let $xN \in HN/N$. Then, $x = hn$ for some $h \in H$ and some $n \in N$ by definition of HN. Since $h^{-1}x = n \in N$, it follows that $xN = hN$.

(c) Define $\varphi : H \to HN/N$. By part (a), HN/N is a group. Moreover, φ is a homomorphism since, given $h, h' \in H$,

$$\varphi(hh') = hh'N = hNh'N = \varphi(h)\varphi(h').$$

Finally, part (b) implies that φ is surjective.

(d) We show that $\ker\varphi = H \cap N$. We clearly have $H \cap N \subseteq \ker\varphi$ since, if $h \in H \cap N$, then $h \in N$, so $\varphi(h) = hN = N$. On the otherhand, the other inclusion is equally obvious. If $h \in \ker\varphi$, then $\varphi(h) = hN = N$, so $h \in N$. But, we already had $h \in H$, so $h \in H \cap N$.

Now, by the fundamental homomorphism theorem, $HN/N \equiv H/\ker\varphi = H/H \cap N$. ∎

28. Let H and N be normal subgroups of G with $N \subseteq H$. Define $\varphi : G/N \to G/H$ by $\varphi(xN) = xH$ for all $xN \in G/N$.

(a) Show that φ is well defined and surjective.

(b) Show that $(G/N)/(H/N) \equiv G/H$.

Proof: (a) Suppose $xN = yN$. Then $y^{-1}x \in N$. But, $N \subseteq H$, so $y^{-1}x \in H$, showing that $yH = xH$. Now, given $xH \in G/H$, observe that $\varphi(xN) = xH$, showing that φ is surjective.

(b) By the fundamental homomorphism theorem, $(G/N)/\ker\varphi \equiv G/H$. Now, if $xN \in \ker\varphi$, then $\varphi(xN) = xH = H$. Thus, $x \in H$, so $xN \in H/N$. It follows that $\ker\varphi \subseteq H/N$. On the otherhand, if $hN \in H/N$, then clearly $\varphi(hN) = H$. Thus $\ker\varphi = H/N$. ∎