Assignment 4: Selected Solutions

Section 3.2

22. Compute the center of $GL_2(\mathbb{R})$.

Proof: Let $G = GL_2(\mathbb{R})$. Note that

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} | a \in \mathbb{R}^\times \right\} \subseteq Z(G).$$

On the otherhand, for $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we have calculated that

$$C(x) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in \mathbb{R}, a \neq 0 \right\}$$

and

$$C(x^t) = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} | a, b \in \mathbb{R}, a \neq 0 \right\}$$

where x^t is the transpose of x. Thus, $Z(G) \subseteq C(x) \cap C(x^t) = D$. ▮

27. Find an example of a group G and elements $a, b \in G$ such that a and b each have finite order, but ab does not.

Proof: Try this: Let $A, B \in GL_2(\mathbb{R})$ be

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

▮

Section 3.3

9. This exercise concerns subgroups of $\mathbb{Z} \times \mathbb{Z}$. NOTE: this problem is incorrect as written in the book. I have rewritten this as follows:

(a) Let $D = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} | a = b\}$. Show that C_1 is a subgroup of $\mathbb{Z} \times \mathbb{Z}$.

(b) Let $C_1 = \mathbb{Z} \times \mathbb{Z}$ and for $n \geq 2$, let $C_n = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} | a \equiv b \pmod{n}\}$. Show that C_n is a subgroup of $\mathbb{Z} \times \mathbb{Z}$.

(c) Show that every subgroup of $\mathbb{Z} \times \mathbb{Z}$ that contains C_1 has the form C_n for some positive integer n.

Proof: Parts (a) and (b) are easy.

We prove (c). First, note that $C_n = D + \{0\} \times n\mathbb{Z}$. Indeed, if $(a, b) \in C_n$, then $a \equiv b \pmod{n}$, so $b = a + nq$. Thus, $(a, b) = (a, a) + (0, nq) \in D + \{0\} \times n\mathbb{Z}$. It follows that $C_n \subseteq D + \{0\} \times n\mathbb{Z}$. To show that $D + \{0\} \times n\mathbb{Z} \subseteq C_n$, just note that every element of $D + \{0\} \times n\mathbb{Z}$ is of the form $(a, a + nq)$ for some $a, q \in \mathbb{Z}$. Now, $(a, a + nq) \in C_n$ since $a \equiv a + nq \pmod{n}$.

Now, let $H \leq \mathbb{Z} \times \mathbb{Z}$ be a subgroup containing D. If $(a, b) \in H$, then $(a, b) - (a, a) = (0, b - a) \in H$ since $D \leq H$. Now, the set $K = \{(0, x) \in H\}$ is clearly a subgroup of H. In fact, it is a subgroup of $\{0\} \times \mathbb{Z}$. It follows that $K = \{0\} \times n\mathbb{Z}$ for some $n \geq 1$. Therefore, $C_n = D + \{0\} \times n\mathbb{Z} \leq H$. On the otherhand, given $(a, b) \in H$, write $(a, b) = (a, a) + (0, b - a) \in C_n$. This shows that $H = C_n$. ▮

14. Let G be a finite group, and let H, K be subgroups of G. Prove that

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Proof: We first prove that $H \cap K$ is a subgroup of H and K. To this end, observe that $e \in H$ and $e \in K$, so $e \in H \cap K$. Also, if $a \in H \cap K$, so is a^{-1} because $a, a^{-1} \in H$ and $a, a^{-1} \in K$. Finally, if $a, b \in H \cap K$, then $a, b, ab \in H$ and $a, b, ab \in K$ so $ab \in H \cap K$.

Now, define and equivalence relation on K by $k_1 \sim k_2$ if $k_1 k_2^{-1} \in H \cap K$. Note that the proof that this is an equivalence relation depends on the fact that $H \cap K$ is a subgroup! (see Proposition 3.2.9.) Now, let $[k]$ denote the equivalence class of k relative to $\sim$. Observe that there are $|K|/|H \cap K|$ such equivalence classes (the proof of this is the same as the one in Lagrange's theorem).

Next, we prove that the sets $Hk_1 = \{hk_1 | h \in H\}$ and $Hk_2 = \{hk_2 | h \in H\}$ are equal if, and only if, $k_1 \sim k_2$. Indeed, if $Hk_1 = Hk_2$, then $k_1 = hk_2$ for some $h \in H$. It follows that $k_1 k_2^{-1} = h$. Therefore, $k_1 k_2^{-1} \in H$. On the otherhand, $k_1 k_2^{-1} \in K$. Thus, $k_1 k_2^{-1} \in H \cap K$ and we have proven that $k_1 \sim k_2$. In the other direction, if $k_1 \sim k_2$, then $k_1 k_2^{-1} \in H \cap K$. Therefore, there exists an $h \in H \cap K$ such that $k_1 = hk_2$. Now, $h \in H$, so $Hh = H$ (h just moves the elements of H around!). Thus, $Hk_1 = Hhk_2 = Hk_2$.

Finally, we are in a position to count. Notice that there are $|K|/|H \cap K|$ distinct sets of the form Hk, one for each equivalence class $[k]$. The size of these sets is $|H|$, and

$$HK = \bigcup_{[k]} Hk.$$

Thus,

$$|HK| = \frac{|H||K|}{|H \cap K|}$$

as promised! ∎