

Assignment 1: Selected Solutions

Section 2.3

5. Let $3 \leq m \leq n$. Calculate $\sigma\tau^{-1}$ for $\sigma = (12 \dots m-1)$ and $\tau = (12 \dots m)$ in S_n .

Proof: I'm lazy, so I'll use problem 13 below. We have $\tau = \sigma(m-1, m)$, so $\tau^{-1} = (m-1, m)\sigma^{-1}$.

Thus,

$$\begin{aligned}\sigma\tau^{-1} &= \sigma(m-1, m)\sigma^{-1} \\ &= (\sigma(m-1), \sigma(m)) \\ &= (1, m).\end{aligned}$$

■

10. Let $\sigma \in S_n$ and suppose that σ is written as a product of disjoint cycles. Show that σ is even if, and only if the number of cycles of even length is even. Show that σ is odd if the number of cycles of even length is odd.

Proof: First note that this only makes sense for $n \geq 2$.

We begin by proving that the cycle $(a_1 \dots a_{2n})$ is odd (i.e. can be written as an odd number of transpositions) and the cycle $(a_1 \dots a_{2n+1})$ is even (i.e. can be written as an even number of transpositions).

Indeed, proceed by induction. For the base case, $(a_1) = (a_1 a_2)(a_1 a_2)$ (we assume here that $a_1 \neq a_2$). Now, for the inductive step, assume that for some $n \geq 1$, $(a_1 \dots a_{2n-1})$ is even. It follows that

$$(a_1 \dots a_{2n-1} a_{2n}) = (a_1 \dots a_{2n-1})(a_{2n-1} a_{2n})$$

is odd, and

$$(a_1 \dots a_{2n} a_{2n+1}) = (a_1 \dots a_{2n})(a_{2n} a_{2n+1})$$

is even.

Now, write $\sigma = \varepsilon_1 \dots \varepsilon_m \omega_1 \dots \omega_n$ where the ε_i, ω_j are all pairwise disjoint cycles, the ε_i have even length and the ω_j have odd length ($1 \leq i \leq m, 1 \leq j \leq n$). Then we may write each ε_i as $2m_i + 1$ transpositions and each ω_j as $2n_j$ transpositions for some $m_i, n_j \in \mathbb{Z}$. Therefore, we can write σ as

$$\sum_{i=1}^m (2m_i + 1) + \sum_{j=1}^n 2n_j = 2 \left(\sum_{i=1}^m m_i + \sum_{j=1}^n n_j \right) + m$$

transpositions. This is even if and only if m is even, and odd if and only if m is odd. ■

13. Let $\tau = (12 \dots k) \in S_n$.

(a) Prove that if $\sigma \in S_n$, then $\sigma\tau\sigma^{-1} = (\sigma(1)\sigma(2) \dots \sigma(k))$.

(b) Let ρ be any cycle of length k . Show that there exists a permutation $\sigma \in S_n$ such that $\sigma\tau\sigma^{-1} = \rho$.

Proof: For (a), we calculate the effects of $\sigma\tau\sigma^{-1}$ on $\sigma(i)$ for $1 \leq i \leq k$. For $i < k$,

$$\sigma\tau\sigma^{-1}(\sigma(i)) = \sigma\tau(i) = \sigma(i+1)$$

and

$$\sigma\tau\sigma^{-1}(\sigma(k)) = \sigma\tau(k) = \sigma(1).$$

Thus, $\sigma\tau\sigma^{-1} = (\sigma(1)\sigma(2)\dots\sigma(k))$.

For (b), let $\rho = (a_1 \dots a_k)$. Denote the elements of $\{1, \dots, n\} \setminus \{a_1, \dots, a_k\}$ by a_{k+1}, \dots, a_n . Define σ by $\sigma(i) = a_i$ for $1 \leq i \leq n$. Then $\sigma \in S_n$ and, by part (a), $\sigma\tau\sigma^{-1} = \rho$. ■

Section 3.1

21. Let G be a finite set with an associative, binary operation given by a table in which each element of the set G appears exactly once in each row and column. Show that G is a group. How do you recognize the identity element? How do you recognize the inverse of an element?

Proof: First, we understand that G is closed under the binary operation since the product gh of two elements $g, h \in G$ is given by the element of G in the row labelled by g and the column labelled by h . Associativity of this operation is given.

Now, let $g, h \in G$. Since h appears exactly once in the row labelled by g , the equation $gx = h$ has a unique solution. Since h appears exactly once in the column labelled by g , the equation $xg = h$ has a unique solution. It follows that G is a group.

One finds the identity element $e \in G$ by solving the equation $gx = g$ for any choice of $g \in G$. The inverse of an element $g \in G$ is found by solving $gx = e$. ■

24. Show that if G is a finite group with an even number of elements, then there must exist an element $a \in G$ with $a \neq e$ and $a^2 = e$.

Proof: Assume that G has an even number of elements. The set $S = \{g \in G \mid g \neq g^{-1}\} = \{g \in G \mid g^2 \neq e\}$ has an even number of elements since if $g \in S$, so is g^{-1} . On the other hand, $G \setminus \{e\}$ has an odd number of elements, so the set $G \setminus (S \cup \{e\}) = \{g \in G \mid g \neq e \text{ and } g^2 = e\}$ is nonempty. ■