A limit of the form 1^∞

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Here's an example to show that a limit of the form 1^∞ might be neither 1 nor infinity.

Theorem 1. *The limit*

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

is greater than 2 *and is less than* 3.

Proof. We use the binomial theorem. Recall that

$$\begin{aligned}(1+a)^2 &= 1+2a+a^2\\ (1+a)^3 &= 1+3a+3a^2+a^3\\ (1+a)^4 &= 1+4a+6a^2+4a^3+a^4.\end{aligned}$$

In general, if n is an integer one can use Pascal's triangle to show that

$$(1+a)^n = 1+\binom{n}{1}a+\binom{n}{2}a^2+\cdots+\binom{n}{n-2}a^{n-2}+\binom{n}{n-1}a^{n-1}+a^n,$$

where

$$\binom{n}{r}=\frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!}.$$

If we apply this formula to the function $(1+\frac{1}{n})^n$, then we get

$$(*)\qquad \left(1+\frac{1}{n}\right)^n = 1+\binom{n}{1}\frac{1}{n}+\binom{n}{2}\frac{1}{n^2}+\cdots+\binom{n}{n-2}\frac{1}{n^{n-2}}+\binom{n}{n-1}\frac{1}{n^{n-1}}+\frac{1}{n^n}.$$

As n gets large the sum gets longer and longer, so in the limit the sum has an infinite number of terms. We don't really know how to deal with infinite sums (this topic is covered extensively in Math 1b), but let's just assume for the moment that we can work with them just like ordinary finite sums.

Now let's consider what happens to each individual term in the sum labeled (*) as n gets large. We want to find

$$\lim_{n\to\infty}\binom{n}{r}\frac{1}{n^r}.$$

By definition of the binomial coefficients $\binom{n}{r}$, this is equal to

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!n^r}.$$

The numerator has r terms, and the denominator has r powers of n, so we may distribute one n to each term:

$$\lim_{n\to\infty} \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)}{r!}.$$

Now, as n gets large, fractions of the form $\frac{a}{n}$ go to zero, so the limit of the numerator is 1. The denominator is $r!$, which doesn't depend on n. Thus we have

$$\lim_{n\to\infty} \binom{n}{r}\frac{1}{n^r} = \frac{1}{r!}.$$

Now we can plug this result back into equation labeled (*), and we see that

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n &= 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}\cdots \\ &= 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\cdots,\end{aligned}$$

where the ... at the end means that the sum goes on forever.

Since all the terms of the sum are positive and the first three are $1+1+1/2$, it is clear that the infinite sum is greater than 2. Furthermore, if we remember that

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 2,$$

then we can match up the terms from our series in question as follows:

$$\frac{1}{1!}\leq 1, \quad \frac{1}{2!}\leq\frac{1}{2}, \quad \frac{1}{3!}<\frac{1}{4}$$

and in general,

$$\frac{1}{r!}\leq\frac{1}{2^r},$$

with strict inequality for $r \geq 3$. We conclude that

$$\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots\right) < \left(1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right) = 3.$$

Thus the infinite sum is less than 3. □

We have just shown that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

is between 2 and 3. In fact the limit is $e \approx 2.7182818$. We will be able to compute the limit precisely once we have learned about derivatives.