Math 1a – Extra Credit Solutions

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1. A solution to the equation $x^2 = 2^x$ is the same as a zero of the function $f(x) = 2^x - x^2$. We will first use the Intermediate Value Theorem to show that $f(x)$ has at least three zeroes. Then we will use Rolle's Theorem to show that $f(x)$ has at most three zeroes. We can use these theorems because $f(x)$ is continuous and infinitely differentiable (that is, we can take as many derivatives of f as we want) for all x.

Two (semi) obvious zeroes of f are $x = 2$ and $x = 4$. Furthermore, $f(0) = 1$ and $f(-1) = -1/2$, so by the Intermediate Value Theorem f must have a zero in the interval $(-1, 0)$. Thus we see that f has at least three zeroes.

Rolle's Theorem says that if g is continuous and differentiable, between any two zeroes of g there is a zero of g'. We will apply the theorem to various derivatives of f. We first calculate these derivatives:

$$\begin{aligned} f(x) &= 2^x - x^2 \\ f'(x) &= 2^x \ln 2 - 2x \\ f''(x) &= 2^x (\ln 2)^2 - 2 \\ f'''(x) &= 2^x (\ln 2)^3 \end{aligned}$$

Now we use Rolle's Theorem. Suppose $f(x)$ had at least four zeroes. Then between every pair of consecutive zeroes of f there would be a zero of f', so f' would have at least three zeroes. Furthermore, between every pair of zeroes of f' there would be a zero of f'', so f'' would have at least two zeroes. Finally, between the two zeroes of f'' there would be a zero of f''', so f''' would have at least one zero.

However, the function 2^x is positive everywhere and $\ln 2$ is a positive constant, so we see that $f'''(x)$ is positive everywhere and therefore has no zeroes. We conclude that $f(x)$ does not have four zeroes, so it must have at most three.

Since $f(x)$ has at least three zeroes and at most three zeroes, it must have exactly three, so there are exactly three values of x for which $x^2 = 2^x$. $\square$

2. Let $V(r)$ be the volume of a sphere of radius r, and let $A(r)$ be the surface area of a sphere of radius r. We are given that $V(r) = \frac{4}{3}\pi r^3$, and we want to calculate $A(r)$.

Solution 1: Consider two concentric spherical shells, one of radius r and the other of radius $r + \Delta r$. The volume between the two shells is$V(r + \Delta r) - V(r)$

As Δr gets very small, the spheres get closer together, and when we zoom in to the scale of Δr, the spheres look very flat. When things are flat, we can use our ordinary formula for volume, which is base area times height. Thus the volume of the region between the spheres is approximately $A(r)\Delta r$. In the limit as Δr goes to zero, this becomes exact:

$$A(r) = \lim_{\Delta r \to 0} \frac{V(r + \Delta r) - V(r)}{\Delta r} = V'(r)$$

by the definition of the derivative. Thus $A(r) = V'(r) = 4\pi r^2$.

Solution 2: In class we calculated the volume of a sphere by dividing the sphere up into discs and adding up the volume of all the small discs. We now use the same idea, but divide up the sphere in a different way. Namely, we divide up the sphere into concentric spherical shells. By the same argument as above, the volume of a shell of radius x and thickness Δx is approximately $A(x)\Delta x$. When we take $\Delta x \to 0$ and add up all the shells from radius zero to r, we get

$$V(r) = \int_0^r A(x)\,dx = \frac{4}{3}\pi r^3$$

By the Fundamental Theorem of Calculus, $V'(r) = A(r) = 4\pi r^2$. □

3. Let $f(x) = (e^{ix} + e^{-ix})/2$ and $g(x) = (e^{ix} - e^{-ix})/2i$. We start by taking derivatives, and use the fact that $i^2 = -1$.

$$\begin{aligned}
f(x) &= \frac{e^{ix} + e^{-ix}}{2} \\
f'(x) &= \frac{ie^{ix} - ie^{-ix}}{2} \\
f''(x) &= \frac{-e^{ix} - e^{-ix}}{2} = -f(x) \\
g(x) &= \frac{e^{ix} - e^{-ix}}{2} \\
g'(x) &= \frac{ie^{ix} + ie^{-ix}}{2} \\
g''(x) &= \frac{-e^{ix} + e^{-ix}}{2} = -g(x)
\end{aligned}$$

From the fact stated in the problem, we now know that $f(x) = A\sin(x) + B\cos x$ and $g(x) = C\sin x + D$ for some constants A, B, C, D. To figure out what these constants are, we compute values at zero:

First, $f(0) = A\sin(0) + B\cos(0) = B$, and $f(0) = (e^0 + e^0)/2 = 1$. Thus $B = 1$. Then, $f'(0) = A\cos(0) - B\sin(0) = A$, and $f'(0) = (ie^0 - ie^0)/2 = 0$, so $A = 0$. We conclude that $f(x) = \cos x$.

Second, $g(0) = C\sin(0) + D\cos(0) = D$, and $g(0) = (e^0 - e^0)/2i = 0$. Thus $D = 0$. Then, $g'(0) = C\cos(0) - D\sin(0) = C$, and $g'(0) = (ie^0 + ie^0)/2i = 1$, so $C = 1$. We conclude that $g(x) = \sin x$. □

4. We need to show that for any $\epsilon > 0$, there is a $\delta > 0$ such that $|e^x - 1| < \epsilon$ whenever $|x| < \delta$. Thus given an ϵ, we need to be able to calculate a δ that works. Our answer will therefore be a formula for δ in terms of ϵ.

We will consider two cases: first, when x is negative, and second, when x is positive. Suppose that x is negative. Since $e^0 = 1$ and e^x is increasing everywhere, $e^x - 1 < 0$ when x is negative, so $|e^x - 1| = 1 - e^x$ for these values of x.

Now, from the inequality $e^x \geq x+1$, multiplying by -1 gives $-e^x \leq -x-1$, and adding 1 gives $1-e^x \leq -x$. Since x is negative, $|x| = -x$. We have therefore shown that $|e^x - 1| \leq |x|$ when x is negative. If we pick $\delta = \epsilon$, then whenever $|x| < \delta$ we have

$$|e^x - 1| \leq |x| < \delta = \epsilon$$

when x is negative. Thus $\delta = \epsilon$ works for negative values of x.

Now suppose that x is positive. Since $e^0 = 1$ and e^x is increasing everywhere, $e^x - 1 > 0$ when x is positive, so $|e^x - 1| = e^x - 1$ for these values of x.

Now from the inequality $e^x < 2x + 1$ for x in $(0, 1)$, subtracting 1 gives $e^x - 1 < 2x$ for these values of x. Furthermore, when x is positive $|x| = x$. We have therefore shown that $|e^x - 1| < 2\,|x|$ whenever x is between 0 and 1. If we pick $\delta = \min\{1, \epsilon/2\}$, then whenever $|x| < \delta$ we have

$$|e^x - 1| \leq 2\,|x| < 2\delta \leq \epsilon$$

.

We have shown that choosing $\delta = \epsilon$ gives us the inequalities we want when x is negative, and choosing $\delta = \min\{1, \epsilon/2\}$ gives us the inequalities we want when x is positive. Thus to find a δ that works for all x we need to choose the smaller of the two:

$$\delta = \min\{1, \epsilon/2, \epsilon\} = \min\{1, \epsilon/2\}.$$

For any ϵ, choosing this value for δ guarantees that $|e^x - 1| < \epsilon$ whenever $|x| < \delta$. $\square$