MATH 53

Quiz 9, November 7th, 2007 Name: ________________________

Answer no more than two of the following questions, indicating clearly which two you would like graded by circling their numbers.

1. Use an appropriate transformation to evaluate $\int\int_R xy\,dA$, where R is the square with vertices $(0,0),(1,1),(2,0)$, and $(1,-1)$.

   Solution: The equations of the lines bounding R are $x+y=2, x+y=0, x-y=2, x-y=0$. So if we make the inverse transformation $u=x+y, v=x-y$, we will get the box $[0,2]\times[0,2]$ in the uv-plane. We then solve for x and y to get the transformation $x=(u+v)/2, y=(u-v)/2$. The Jacobian of this transformation is $\frac{1}{2}$. Also note that $xy=\frac{(u+v)(u-v)}{4}=\frac{u^2-v^2}{4}$. So by the change of variables theorem we have $\int\int_R xy\,dA=\int_0^2\int_0^2(u^2-v^2)\frac{1}{8}\,du\,dv=\frac{1}{8}\int_0^2\left[\frac{u^3}{3}-uv^2\right]_{u=0}^{u=2}dv=\frac{1}{8}\int_0^2\frac{8}{3}-2v^2\,dv=\frac{1}{8}\left[\frac{8}{3}v-\frac{2}{3}v^3\right]_0^2=\frac{1}{8}(\frac{16}{3}-\frac{16}{3})=0$. This agrees with the symmetry of the function xy on the square R.

2. Use the transformation $x=u^2, y=v^2, z=w^2$ to set up an integral to find the volume of the region bounded by the surface $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$ and the coordinate planes. Write the integral as an iterated integral in the order $dw\,dv\,du$.

   Solution: Under this transformation the u,v,w coordinate planes in the first octant get sent to the coordinate planes in the first octant in xyz space. To find out which points (u,v,w) get sent to the surface $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$, we substitute u,v,w into the equation to get $\sqrt{u^2}+\sqrt{v^2}+\sqrt{w^2}=1\Rightarrow u+v+w=1$. Thus the region in uvw space which gets sent to the region is the tetrahedron bounded by the coordinate planes and the plane $u+v+x=1$. The Jacobian of the transformation is $8uvw$. So by the change of variables theorem, we get $\int_0^1\int_0^{1-u}\int_0^{1-u-v}8uvw\,dw\,dv\,du$ (this integral is possible to evaluate).

3. Graph the following vector fields:

   $$\text{(a) } \mathbf{F}(x,y)=\sin(x)\mathbf{i}+y^2\mathbf{j} \qquad \text{(b)} \mathbf{G}(x,y)=-y^2\mathbf{i}+x\mathbf{j}$$

   Show that at least one of them is conservative. (Use the back if necessary.)

   Solution: We show that F is conservative by exhibiting a potential function. Let $h(x,y)=\cos(x)+\frac{1}{3}y^3$. Then $\nabla h(x,y)=\sin(x)\mathbf{i}+y^2\mathbf{j}=\mathbf{F}(x,y)$.