MATH 53
Quiz 11, December 5th, 2007 Name: ________________________

Answer no more than two of the following questions, indicating clearly which two you would like graded by circling their numbers.

1. Let T be the surface obtained by removing the portion of the sphere $\rho = 2$ that lies inside the cone $z^2 = x^2 + y^2$.
   ($T = \{(x, y, z)|x^2 + y^2 + z^2 = 4$ and $z^2 \leq x^2 + y^2\}$) Find the surface area of T.

   Solution: We use the spherical coordinate parametrization which is $\mathbf{r}(\theta, \phi) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle$. We have $|\mathbf{r}_\theta \times \mathbf{r}_\phi| = 4\sin\phi$ (from the result in disscussion). Now to get the portion of the sphere we want, we must restrict ϕ between $\pi/4$ and $3\pi/4$. So the surface area is $\int_0^{2\pi}\int_{\pi/4}^{3\pi/4} 4\sin\phi\, d\phi\, d\theta = 8\pi\left[-\cos\phi\right]_{\pi/4}^{3\pi/4} = 8\sqrt{2}\pi$.

2. Let T be the surface $x = \sqrt{y^2 + z^2}, 0 \leq x \leq 1$ oriented in the negative x direction. Let $\mathbf{F} = \langle x^2, y, z\rangle$. Compute $\int\int_T \mathbf{F} \cdot d\mathbf{S}$ directly.

   Solution: The first task is to parameterize T. There are several possibilities. One is to thing of T as a surface of revolution. Then we have $\mathbf{r}(x, \theta) = \langle x, x\cos\theta, x\sin\theta\rangle, 0 \leq x \leq 1, 0 \leq \theta \leq 2\pi$. Then $\mathbf{r}_x \times \mathbf{r}_\theta = \langle 1, \cos\theta, \sin\theta\rangle \times \langle 0, -x\sin\theta, x\cos\theta\rangle = \left\langle x\cos^2\theta + x\sin^2\theta, -x\cos\theta, -x\sin\theta\right\rangle = \langle x, -x\cos\theta, x\sin\theta\rangle$. Since this normal vector has positive $\mathbf{i}$ component we instead use $\langle -x, x\cos\theta, x\sin\theta\rangle$. We now compute $\int\int_T \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 \langle x^2, x\cos\theta, x\sin\theta\rangle\cdot\langle -x, x\sin\theta, x\cos\theta\rangle\ dx\, d\theta = \int_0^{2\pi}\int_0^1 -x^3 + x^2\sin^2\theta + x^2\cos^2\theta\, dx\, d\theta = \int_0^{2\pi}\int_0^1 -x^3 + x^2\, dx\, d\theta = 2\pi(\frac{1}{3} - \frac{1}{4}) = \frac{\pi}{6}$.

3. Let $\mathbf{F} = \langle -2xz, 6x, y^2\rangle$. Let C be the intersection of $4x^2 + 9y^2 = 1$ with the upper unit hemisphere and C is oriented counterclockwise when viewed from above. Calculate $\int\int_C \mathbf{F} \cdot d\mathbf{r}$. [Hint: Parameterize the sphere as the graph of a function. Also, the area of the ellipse $4x^2 + 9y^2 = 1$ is $\pi/6$.]

   Solution: We use Stokes' Theorem. Let T be the portion of the hemisphere that lies inside the cylinder $4x^2 + 9y^2 = 1$ oriented upward. Then C is the positively oriented boundary of T. So by Stokes Theorem $\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_T \nabla \times \mathbf{F} \cdot d\mathbf{S}$. First we calculate $\nabla \times \mathbf{F} = \langle 2y, -2x, 6\rangle$. Then since T is the graph of the function $g(x, y) = \sqrt{1 - x^2 - y^2}$ with domain D given by $4x^2 + 9y^2 \leq 1$, we have $\int\int_T \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int\int_D -P \cdot g_x - Q \cdot g_y + R\, dA = \int\int_D -2y \cdot \frac{-x}{\sqrt{1-x^2-y^2}} + 2x\frac{-y}{\sqrt{1-x^2-y^2}} + 6\, dA = \int\int_D 6\, dA = \pi$.