MATH 53

Quiz 10, November 20th, 2007 Name: ________________________

Write clearly and justify all work.

1. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \cos x^2\mathbf{i} + (\sin y^2 + \frac{x^3}{3} + xy^2)\mathbf{j}$ and C is the unit circle oriented clockwise.

   Solution: By Green's theorem, this is equal to $-\int\int_D x^2 + y^2 dA$ where D is the unit circle. In polar coordinates we get $\int_0^1 \int_0^{2\pi} r^3 d\theta dr = -\pi/2$.

2. For each of the following functions, calculate the gradient, divergence and curl or indicate that the operation doesn't make sense:

   (a) $f(x,y,z) = x^2 + zy$, (b) $\mathbf{F}(x,y,z) = \nabla(xyz + \sin x \sin y \sin z)$

   (c) $\mathbf{G}(x,y,z) = xy\mathbf{i} + yz\mathbf{k}$

   Are any of them conservative?

   Solution:

   (a) $\nabla f(x,y,z) = 2x\mathbf{i} + x\mathbf{j} + z\mathbf{k}$. The other operations don't make sense.

   (b) $\nabla \cdot \mathbf{F}(x,y,z) = -3\sin x \sin y \sin z$, and $\nabla \times \mathbf{F} = 0$ since the curl of a gradient is always zero. Since $\mathbf{F}$ is the gradient of a function, it is conservative.

   (c) $\nabla \cdot \mathbf{G} = 2y$, $\nabla \times \mathbf{G} = z\mathbf{i} - x\mathbf{k}$. Since the curl of $\mathbf{G}$ is nonzero, $\mathbf{G}$ cannot be the gradient of a function and so is not conservative.

3. Let $\mathbf{F}(x,y,z) = (x \sin y, \cos y, zy)$. Either find a vector function $\mathbf{G}(x,y,z)$ such that $\nabla \times \mathbf{G} = \mathbf{F}$ or explain why no such $\mathbf{G}$ can exist.

   Solution: We first calculate $\nabla \cdot \mathbf{F}$ since if $\mathbf{F}$ is the curl of some function $\mathbf{G}$, then the divergence of $\mathbf{F}$ must be zero. We have $\nabla \cdot \mathbf{F} = \sin y - \sin y + y = y \neq 0$. Thus $\mathbf{F}$ cannot be the curl of another function, so no $\mathbf{G}$ can exist.