MATH 53
Quiz 10, November 20th, 2007 Name: ________________________

Write clearly and justify all work.

1. Let C be the curve that traverses the quarter circle $x^2 + y^2 = 1, y \geq 0, x \geq 0$ counterclockwise and then the line segment (0,1) to (0,0). Compute $\int_C (-2y + x^2)\,dx + \sin y^2\,dy$. [Hint: Don't compute the line integral directly, apply Green's theorem to $\mathbf{F} = \langle -2y + x^2, \sin y^2 \rangle$ and the quarter disc $x^2 + y^2 \leq 1, x \geq 0, y \geq 0$].

   Solution: If we call the quarter disc R, Green's Theorem applied to $\mathbf{F}$ and R gives $\int_{\partial R} (-2y + x^2)\,dx + \sin y^2\,dy = \int\int_R (0 - -2)\,dA = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$. However, if we let C_1 be the line segment from (0,0) to $(1,0)$, we have $\partial R = C + C_1$, so $\int_C (-2y+x^2)\,dx + \sin y^2\,dy = \frac{\pi}{2} - \int_{C_1} (-2y + \sin x^2)\,dx + y^2\,dy$. By parameterizing C_1 as $x = t, y = 0, 0 \leq t \leq 1$, we compute $\int_{C_1} (-2y + x^2)\,dx + \sin y^2\,dy = \int_0^1 \langle t^2, 0 \rangle \cdot \langle 1, 0 \rangle\,dt = \frac{1}{3}$. So $\int_C (-2y + x^2)\,dx + \sin y^2\,dy = \frac{\pi}{2} - \frac{1}{3}$.

2. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle oriented counterclockwise and $\mathbf{F}$ is the following vector field in the plane:

$$\mathbf{F}(x, y) = \langle -y^3 + \sin(\sin x), x^3 + \sin(\sin y) \rangle.$$

   Solution: By Green's theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \int\int_D 3x^2 + 3y^2 dA = 3 \int\int_D x^2 + y^2 dA,$$

   which we may compute in polar coordinates as

$$3 \int\int_D x^2 + y^2 dA = 3 \int_0^{2\pi} \int_0^1 r^3 dr d\theta = \frac{3\pi}{2}.$$