MATH 53
Midterm 2 Review
Solutions to Hutching's '03 Midterm

1. Find the minimum and maximum values of the function

$$f(x, y) = (x-1)^2 + (y-1)^2$$

on the unit disc $x^2 + y^2 \leq 1$.

Solution: First we compute the critical points of f, finding $\nabla f = \mathbf{0}$ when $(x, y) = (1, 1)$, which is not a point in the unit disc. Thus, the minima and maxima occur on the boundary of the disc. We apply the method of Lagrange multipliers with $g(x, y) = x^2 + y^2$, obtaining equations

$$2(x-1) = 2\lambda x, \quad 2(y-1) = 2\lambda y, \quad x^2 + y^2 = 1.$$

The first two equations show $x = y$, so by the second we find that $x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}$. By plugging these into f, we find a max at $(-\sqrt{2}, -\sqrt{2})$ and a min at $(\sqrt{2}, \sqrt{2})$.

2. Calculate

$$\int_0^1 \int_{x^{2/3}}^1 x\cos(y^4)\,dydx.$$

Solution: Call the value of the integral I. Then switching the order of integration, we find

$$I = \int_0^1 \int_0^{y^{3/2}} x\cos(y^4)\,dxdy = \int_0^1 \frac{y^3}{2}\cos y^4 dy = \frac{\sin 1}{8}$$

where the last equality is gotten by u-substitution.

3. Calculate

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)^{2003} dxdy.$$

Solution: This integral is over the quarter circle in the first quadrant, so changing to polar coordinates

$$I = \int_0^{\pi/2} \int_0^1 (r^2)^{2003} r\,drd\theta = \frac{\pi}{2}\int_0^1 r^{4007} dr = \frac{\pi}{8016}.$$

4. Find the area enclosed by $x^2 + xy + y^2 = 1$ using the substitution $x = u + v\sqrt{3}$ and $y = u - v\sqrt{3}$.

Solution: Under the substitution, the curve becomes $u^2 + v^2 = 1/3$, so the transformed region (call it S) is the circle of radius $1/\sqrt{3}$. We find the Jacobian to be $-2\sqrt{3}$. Thus,

$$I = \int\int_S |-2\sqrt{3}|dA = 2\sqrt{3}\pi/3 = \frac{2\pi}{\sqrt{3}}.$$

5. Let C be a plane curve starting at $(0, 0)$ and ending at $(1, 1)$. Let

$$\mathbf{F}(x, y) = \langle x^2 + y, y^2 + x \rangle.$$

Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for every C, and compute the value.

Solution: Note that $\mathbf{F} = \nabla(x^3/3 + y^3/3 + xy)$, so as a conservative vector field, $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path-independent. Let $\mathbf{r}(t) = (t, t)$, the straight line from $(0, 0)$ to $(1, 1)$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F} \cdot (1, 1) dt = \int_0^1 2(t^2 + t) dt = \frac{2}{3} + 1 = \frac{5}{3}.$$

Or using the fundamental theorem of line integrals,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(0, 0) = 1/3 + 1/3 + 1 = 5/3.$$