Math 53 Practice Midterm Solutions
For the exam given 4/10/07; Nate Watson

1. Find the volume of the solid region between the surfaces $z = 2x^2+2y^2$ and $z = 12-x^2-y^2$.

Rewrite the equations in polar coordinates as $z = 2r^2$ and $z = 12 - r^2$. We can solve these equations to find the curve of intersection, we find that it is given by $z = 8$ and $r^2 = 4$ so the surfaces intersect on the circle of radius 2 with z coordinate 8. The region we want to find the volume of therefore projects onto the disc $r < 2$. So we integrate the difference of the two functions that the surfaces graph over this region using polar integration to find the volume between them:

$$V = \int_0^{2\pi} \int_0^2 ((12 - r^2) - 2r^2)r \; dr \; d\theta = 24\pi$$

2. Find the maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 + 5y$$

on the region $x^2 + y^2 \leq 4$, and say where the function takes these values

The minimum/maximum are either local minimum or maxima in the interior, in which case they are at critical points of $f(x, y)$, or they lie on the boundary, in which case they are minima/maxima of f subject to the constraint of being on the boundary, that is, the constraint $x^2 + y^2 = 4$. So we can find the minimum/maximimums by checking the value of $f(x, y)$ at all critical points and at the candidate points on the boundary given to us by the method of Lagrange multipliers.
$f_x(x, y) = 2x$ and $f_y(x, y) = 2y + 5$ so these partial derivatives are simultaneously 0 only at the point $(0, -5/2)$, which isn't in the region. So there aren't any critical points to check. On the boundary we use the method of Lagrange multipliers with constraint $g(x, y) = x^2 + y^2 = 4$ and get the equations $f_x(x, y) = \lambda g_x(x, y)$ or $2x = \lambda 2x$ and also $f_y(x, y) = \lambda g_y(x, y)$ or $2y+5 = \lambda 2y$. So either $x = 0$ in which case $y = \pm 2$ from the constraint, giving $f(0, -2) = -6$ and $f(0, 2) = 14$ or $x \neq 0$ and $\lambda = 1$, which is ruled out by the second equation. So the maximum of $f(x, y)$ on this region is 14, which is achieved at $(0, 2)$ and the minimum is -6 which is achieved at $(0, -2)$.

3. Evaluate the iterated integral

$$\int_0^1 \int_x^1 \frac{\cos y}{y} \; dy \; dx$$

The integrand on the inside looks tough, so change the order! You can hopefully check that we are integrating over a triangle with vertices $(0, 0), (0, 1)$, and $(1, 1)$. So set it up the other way, you get

$$\int_0^1 \int_0^y \frac{\cos y}{y} \; dx \; dy = \int_0^1 \cos y \; dy = \sin(1)$$

4. Evaluate the triple integral

$$\iiint_E (x^2 + y^2 + z^2)^{3/2} \; dV$$

where E is the region determined by the inequalities $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$, and $z^2 \leq x^2 + y^2$.

The nastyness of this setup strikes enough fear into our heart to inspire us to find an alternative to doing it in rectangular coordinates. Notice that the inequality $x^2 + y^2 + z^2 \leq 1$ describes the region in a sphere of radius 1. The other two say we want the part of this sphere that is above the xy-plane and below the cone described by $r = |z|$ in polar coordinates. We know in spherical coordinates the plane $z = 0$ is described by $\phi = \pi/2$ and $r = |z|$ becomes $\rho \sin \phi = |\rho \cos \phi|$ or $\tan \phi = \pm\pi/4$, of which we only care about the part above the xy-plane, which is $\phi = \pi/4$. Finally the integrand is $(\rho^2)^{3/2} = |\rho|^3 = \rho^3$ so (not forgeting the spherical volume element) we get the integral

$$\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^1 \rho^3(\rho^2 \sin \phi)\ d\rho\ d\phi\ d\theta = \frac{\sqrt{2}\pi}{6}$$

5. Let R denote the triangle in the xy-plane with corners at $(0, 0)$, $(1, 0)$ and $(0, 1)$. Use the change of variables $x = u^2$, $y = v^2$ to evaluate the double integral

$$\iint\limits_R \frac{1}{\sqrt{xy}}\ dA.$$

The region in the xy-plane is between the curves $x = 0$ and $y = 0$ and $x + y = 1$. So we must find what the image of the region is in the uv-plane, or rather, what is the region in the uv-plane which is taken to the region R by our transformation. Well, $x = 0$ corresponds to $u^2 = 0$ that is $u = 0$ and $y = 0$ corresponds to $v = 0$. $x + y = 0$ corresponds to $u^2 + v^2 = 1$, which is the unit circle. So the corresponding region S in the uv plane is the region inside the unit disk in the first quadrant. The integrand becomes $\frac{1}{\sqrt{xy}} = \frac{1}{uv}$ using the fact that u and v are positive in the region we are worried about. The Jacobian is $\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = 4uv$. So our answer is

$$\iint\limits_R \frac{1}{\sqrt{xy}}\ dA = \iint\limits_S \frac{1}{uv}|4uv|\ dA = 4 \iint\limits_S\ dA.$$

We can evaluate this last integral by doing a polar double integral or by just noticing it's the area of S, which being a fourth of a unit circle, is $\pi/4$. So our answer is $\iint\limits_R \frac{1}{\sqrt{xy}} = \pi$.

6. Evaluate the iterated integral

$$\int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}} e^{x^2+y^2}\ dy\ dx$$

If we sketch the region we are integrating over, it is the "pie piece" of the unit circle between the line $x = 0$ and $y = x$. Set this up as a polar integral. The integrand becomes e^r. Don't forget the polar area element. You get

$$\int_{\pi/4}^{\pi/2} \int_0^1 e^r r\ dr\ d\theta = \pi/4.$$

Note that the inside integral can be evaluated using integration by parts.