Monday, November 6 2007
Math 53

1. Let $\mathbf{F} = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$.

   (a) Compute $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. *Solution:* 0

   (b) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle traversed counter-clockwise. *Solution:* 2π.

   (c) Is $\mathbf{F}$ conservative? *Solution:* No, since the line integral around along a closed curve is not zero. This can happen despite $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ since that equality does not hold on a simply connected region ($\mathbf{R}^2 - \{(0,0)\}$ is not simply connected).

   (d) What if C does not enclose the origin? *Solution:* Then we can draw a simply connected region R that does not contain the origin but contains C. Then $\mathbf{F}$ is conservative on R so the line integral along C will be zero.

2. Let $\mathbf{F} = \langle P, Q, R \rangle$. Write out what it means for $\mathbf{F} = \nabla f$ in terms of the component functions of $\mathbf{F}$. Then compute $\nabla \times \mathbf{F}$. *Solution:* $\mathbf{F} = \langle f_x, f_y, f_z \rangle$. Then $\nabla \times \mathbf{F} = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{xy} - f_{xy} \rangle = \mathbf{0}$ by clairaut's theorem.

3. Again let $\mathbf{F} = \langle P, Q, R\rangle$. Write out what it means for $\mathbf{F} = \nabla \times \mathbf{G}$. Then compute $\nabla\cdot\mathbf{F}$. *Solution:* Let $\mathbf{G} = \langle S, T, U\rangle$. Then $\mathbf{F} = \langle U_y - T_z, S_z - U_x, T_x - S_y\rangle$. So $\nabla\cdot\mathbf{F} = (U_{yx} - T_{zx}) + (S_{zy} - U_{xy}) + (T_{xz} - S_{yz}) = 0$ (again by Clairaut's theorem).

4. Let $\mathbf{F} = \langle 1 + ze^y, xze^y, xe^y\rangle$.

   (a) Can $\mathbf{F}$ be the gradient of some function $f(x, y, z)$? *Solution:* Yes, since by computation $\nabla \times \mathbf{F} = \mathbf{0}$.

   (b) If the answer to part (a) is "yes," find all functions $f(x, y, z)$ of which $\mathbf{F}$ is the gradient. *Solution:* $f(x, y, z) = x + xze^y + k$.