

# Aspects of Cylindrical Symmetry in Gauge Theory Gravity

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# Chapter 1

## Introduction and Basic Formalism

Exact solutions in General Relativity are useful not only in the extent to which they resemble real astrophysical phenomena, but also in how they probe limitations and quirks of the theory. In this work we aim to understand the restrictions placed upon fields with two-dimensional spatial symmetry. All solutions found have been discovered in various settings by other authors so the utility of our approach lies not as much in the production of solutions, but rather in the ability to form a catalogue of what gravity does and does not allow given particular physical circumstances. Gauge Theory Gravity (GTG) helps keep our assumptions clear, while also simplifying the mathematics so that we may address many solutions in succession.

In this way, we also look to demonstrate the power the “intrinsic method” [18] and GTG wield in streamlining solution finding. There are the standard advantages GTG offers over GR, which we briefly mention here. For more thorough discussion, see [7, 9, 18, 27]. First and foremost, while traditional GR fixes rotational freedom at the metric level, GTG forms a set of displacement-gauge covariant intrinsic equations. It is unsurprising that via judicious gauge choices one usually produces a much simpler set of field equations. Indeed, traditional GR will only produce equations as simple as those got by the intrinsic method in the luckiest of circumstances. Second, the existence of a flat background in GTG does away with the many of the confusing aspects of differential geometry in GR. As we will see, the notion of elementary flatness in cylindrical systems is a classic example of an intuitive notion captured much better in GTG. Third, all physically relevant quantities are unambiguous—they must be gauge invariant. This becomes particularly useful in our discussion of intrinsic fluid velocities. And lastly, although tetrad and Newman-Penrose (NP) formalism are extremely similar in spirit to GTG (and indeed share many of the above stated strengths) it is the language of Geometric Algebra (GA) in which GTG is couched that makes the theory so mathematically elegant and computationally efficient.

In this work we wish to focus and enhance the work of [18], and [22] in utilizing Petrov classification and restrictions on the Weyl tensor to find solutions. Such a task is particularly well suited to systems with symmetry, as often such fields are algebraically special [26]. Although our approach is very similar in

spirit to the NP formalism, the language of GA seems much more natural at a very fundamental level. For example, whereas the NP formalism requires a completely unmotivated and somewhat bizarre formal complexification of space, GTG identifies this mathematical artifact as arising from a geometric entity: the pseudoscalar.

Fields with a two-dimensional symmetry group with two-dimensional space-like orbits (“a  $G_2$  on  $S_2$ ”) provide sufficiently simple equations that solution-finding is tractable; however, the behavior of gravity is still sufficiently complicated under these restrictions to produce interesting phenomena. As a point of contrast, the variety of spherically symmetric vacuum fields is constrained by Birkoff’s theorem. Cylindrical- and planar-symmetric fields have no such restrictions, allowing for gravitational waves which we examine in chapter 4. For non-empty spacetimes,  $G_2$  cosmologies are a popular and simple model for inhomogeneities in the early universe [30], and many interesting astrophysical systems admit cylindrical symmetry. Indeed, time-dependent rotating dust is the focus of chapters 2 and 3 where we begin to provide a counterpart for the work done on stationary cylindrically symmetric field in GTG [9, 27].

In the remainder of this chapter we wish to lay the groundwork for our approach and make some general observations about the effects of two-dimensional symmetry on gravitational fields. We begin by reviewing the bare necessities of GA and GTG before translating Petrov’s classification of the Weyl tensor into the form most useful in our analysis, reviewing both its physical and mathematical content. Then we provide the general forms for time-dependent gauge fields with a  $G_2$ . We discuss gauge freedom, as well as other features and bounds of the subsequent class of solutions we develop, including the bracket structure, curvature, and our assumptions on the stress energy tensor. At this stage, the intrinsic method allows us to state many features of solutions, which helps guide our assumptions in what follows.

In Chapter 2 we follow in the steps of [27] and investigate a column of rigidly rotating dust. However, unlike the wonderfully complicated and (at times) pathological stationary van Stockum solution [29], we find that absolutely no interesting features exist when time-dependence is added; in fact, the solution is found to be a Friedman-Robertson-Walker (FRW) cosmological model. In the stationary case, it was shown that an intrinsic velocity determined the gauge sector of the solution (low, critical and high mass regimes) [27]; with time-dependence, this velocity is shown to vanish, having profound implications for the construction of rigidly rotating dust cylinders.

In Chapter 3 we analyze dust which has vanishing “intrinsic” fluid velocity, which corresponds precisely to fields whose Weyl tensor has pure Coulombic and transverse wave forces. With no localized sources of gravitational waves these are the “silent universe” models as described in [19]. We find all solutions for vacuum and dust stress-energy tensors, showing that two-dimensional symmetry in silent universes implies the existence of a three-dimensional (or larger) symmetry group. Therefore it is seen that vanishing rotation increases the symmetry of the fields, and that solutions admitting a maximal  $G_2$  must have nonvanishing intrinsic fluid velocity. Given these results, we outline a technique of solution-finding for general cylindrically symmetric systems by placing restrictions on the intrinsic velocity and shear.

Chapter 4 breaks from our analysis of rotating dust, and instead we consider algebraically special vacuum and pure radiation spacetimes. We show the fields

to yield the class of gravitational plane wave solutions and prove the Goldberg–Sachs theorem for this restricted case with symmetry. Considering a few special solutions, we rephrase many of the typical notions of gravitational waves in the language of GTG and suggest possible inroads to axisymmetric waves and colliding plane waves.

## 1.1 Geometric and Spacetime Algebra

By now there exist many excellent and complete introductions to GA and STA [7, 14, 15]. In the following we merely state results we plan to use and establish notation.

A *geometric algebra* is a graded vector space with an associative product that is distributive over addition and has the property that the square of any vector (a grade–1 element) is a scalar (a grade–0 element). This final axiom is critical, for observe that for vectors  $a$  and  $b$ ,

$$(a + b)^2 = a^2 + ab + ba + b^2. \quad (1.1)$$

Therefore the symmetrized product of two vectors must be a scalar, prompting the definition of an inner product

$$a \cdot b = \frac{1}{2}(ab + ba). \quad (1.2)$$

Similarly, the antisymmetrized product yields a pure bivector, defining an outer product

$$a \wedge b = \frac{1}{2}(ab - ba). \quad (1.3)$$

We interpret  $a \wedge b$  as a directed 2-dimensional volume. This picture generalizes to any  $r$ -dimensional volume and pure grade– $r$  element, which we call a grade– $r$  blade. Putting this together, one establishes<sup>1</sup>

$$ab = a \cdot b + a \wedge b. \quad (1.4)$$

A general element of the algebra is referred to as a multivector. That the geometric product mixes different grades makes an extremely computationally efficient mathematical language. In particular, it is this property that makes the geometric product invertible (in euclidean spaces) whereas the inner and outer products alone are not; e.g. for vectors,

$$a^{-1} = a/a^2. \quad (1.5)$$

This distinguishes GA from other standard theories of inner product spaces or the algebra of differential forms.

In addition to the product, geometric algebras are endowed with several other operations we will need. The first is reversal, which simply reverses the order of a product of vectors

$$(ab \dots c)^\dagger = c \dots ba. \quad (1.6)$$

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<sup>1</sup>To minimize notational clutter, we always assume that inner and outer products are evaluated before geometric products, e.g.  $a \cdot bc$  is interpreted as  $(a \cdot b)c$ .

We extend the operation to arbitrary multivectors by linearity. There is also the commutator product,

$$A \times B = \frac{1}{2}(AB - BA), \quad (1.7)$$

for multivectors  $A$  and  $B$ , which is necessarily antisymmetric as the name suggests. This product satisfies the Jacobi identity and is closed on the space of bivectors, a fact of utmost importance in the study of rotations. In GA, rotors are defined as being scalar + bivector combinations satisfying

$$RR^\dagger = 1, \quad (1.8)$$

and describe rotations of vectors by

$$a \mapsto Ra\tilde{R}. \quad (1.9)$$

This extends to multivectors as

$$ab \dots c \mapsto Ra\tilde{R}Rb\tilde{R} \dots Rc\tilde{R} = R(ab \dots c)\tilde{R}, \quad (1.10)$$

and linearity over addition. In Euclidean spaces we may write

$$R = \exp(-B/2) \quad (1.11)$$

for a bivector  $B$ , so that the space of bivectors with the Lie product,  $\times$ , forms a Lie algebra for the space of rotors, and thus any bivector (or plane) can be thought of as an infinitesimal generator of rotations within that plane. We extend the definition of a linear function,  $\underline{F}$ , acting on vectors to one acting on multivectors by

$$\underline{F}(a \wedge b \wedge \dots \wedge c) = \underline{F}(a) \wedge \underline{F}(b) \wedge \dots \wedge \underline{F}(c), \quad (1.12)$$

combined with linearity over sums of grade- $r$  blades. The adjoint,  $\overline{F}$ , is given by

$$\underline{F}(a) \cdot b = a \cdot \overline{F}(b). \quad (1.13)$$

The psuedoscalar,  $I$ , is defined (up to magnitude) as being the highest grade element of a geometric algebra. Amongst useful roles in complex analysis and Hodge-duality, the psuedoscalar also allows

$$\det(\underline{F})I = \underline{F}(I). \quad (1.14)$$

It can be shown that this agrees with the matrix definition of the determinant, though 1.14 is far more compact and makes obvious the connection to distortions of space.

Lastly we wish to introduce the vector derivative from Geometric Calculus, which we define in terms of a directional derivative

$$b \cdot \partial_a F(a) = \lim_{\epsilon \rightarrow 0} \frac{F(a + \epsilon b) - F(a)}{\epsilon}. \quad (1.15)$$

In a frame  $\{e_i\}$  with reciprocal frame  $\{e^j\}$ ,  $e_i e^j = \delta_i^j$ , we find

$$\partial_a = e^k e_k \cdot \partial_a, \quad (1.16)$$

with the summation convention in effect. From this it is clear that the vector derivative has the properties of a vector. Because of wide-ranging utility, the vector derivative with respect to spatial position will be denoted  $\nabla$ . Along with its myriad uses in multivariable calculus, the vector derivative allows for simple frame-free expressions for contractions, e.g.

$$\text{Tr}(f(a)) = \partial_a \cdot f(a). \quad (1.17)$$

Perhaps the most natural language in which to study relativity is that of the spacetime algebra (STA) [14], a geometric algebra on the vector space defined by

$$\gamma_\mu \cdot \gamma_\nu = \alpha_{\mu\nu} = \text{diag}(+ - - -). \quad (1.18)$$

The full STA is a 16 dimensional vector space spanned by

$$\begin{array}{cccccc} 1, & \{\gamma_\mu\}, & \{\sigma_\mu, I\sigma_\mu\} & \{I\gamma_\mu\} & I & \\ 1 \text{ scalar} & 4 \text{ vectors} & 6 \text{ bivectors} & 4 \text{ trivectors} & 1 \text{ pseudoscalar}, & \end{array} \quad (1.19)$$

where  $\sigma_\mu = \gamma_\mu \gamma_0$ , and the magnitude and orientation of the pseudoscalar is defined by

$$I = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \sigma_1 \sigma_2 \sigma_3. \quad (1.20)$$

The vector  $\gamma_0$  gives a spacetime split, wherein the spatial vector  $\mathbf{a}$  associated to a spacetime vector  $a$  is defined by

$$a\gamma_0 = a_0 + \mathbf{a}, \quad (1.21)$$

where  $a_0$  is a scalar,  $\gamma_0 \cdot a$ , and  $\mathbf{a}$  is a bivector,  $a \wedge \gamma_0$ . It is clear that the three bivectors,  $\sigma_\mu$ , form an orthonormal frame for the spatial vectors. These definitions work equally well with an arbitrary timelike vector  $v$  rather than  $\gamma_0$ .

## 1.2 Gauge Theory Gravity and the Intrinsic Method

At its core, a gauge theoretic description of gravity must maintain that physical relations of fields are invariant under arbitrary local displacements and rotations of the fields. This idea has long been held by various authors who have had limited success in creating a theory of this ilk [11, 17, 28]. However, it is the language of GA that has allowed more robust interpretation and implementation of these intuitive physical properties, culminating in GTG.

In an effort to satisfy the above physical observation, GTG requires two fields, one that ensures covariance under local displacements—a vector-valued displacement-gauge field,  $\bar{h}$ —and one for covariance under local rotations—a bivector-valued rotation gauge field,  $\Omega$ . Both depend on spacetime position  $x$  and a vector  $a$ , being linear in the latter. Under local displacements,  $x \mapsto f(x)$ , the fields transform as

$$\bar{h}(a, x) \mapsto \bar{h}(\bar{f}^{-1}(a), f(x)), \quad \Omega(a, x) \mapsto \Omega(a, f(x))\underline{f}(a). \quad (1.22)$$

Under local rotations parametrized by a rotor  $R = R(x)$ ,

$$\bar{h}(a, x) \mapsto R\bar{h}(a, x)\tilde{R} \quad (1.23)$$

$$\Omega(a, x) \mapsto R\Omega(\tilde{R}aR, x)\tilde{R} - 2L_{\tilde{R}aR}R\tilde{R}, \quad (1.24)$$

where

$$L_a = a \cdot \bar{h}(\nabla). \quad (1.25)$$

A fully covariant directional derivative of a multivector  $M$  is given by

$$a \cdot \mathcal{D}M = L_a M + \omega(a) \times M, \quad (1.26)$$

where we define the displacement-gauge covariant field

$$\omega(a, x) = \Omega(a, x) \underline{h}(a, x), \quad (1.27)$$

which transforms under displacements and rotations as

$$\omega(a, x) \mapsto \omega(a, f(x)), \quad \omega(a, x) \mapsto R\omega(\tilde{R}aR, x)\tilde{R} - 2L_{\tilde{R}aR}R\tilde{R}, \quad (1.28)$$

respectively. Equation 1.26 in turn defines the covariant vector derivative

$$\mathcal{D}M = \mathcal{D} \cdot M + \mathcal{D} \wedge M \quad (1.29)$$

where

$$\mathcal{D} \cdot M = \partial_a \cdot (a \cdot \mathcal{D}M), \quad \mathcal{D} \wedge M = \partial_a \wedge (a \cdot \mathcal{D}M). \quad (1.30)$$

It can be shown that the commutators of the  $L_a$  derivatives satisfy the Jacobi identity and are determined by

$$[L_a, L_b] = L_c, \quad (1.31)$$

where

$$c = a \cdot \mathcal{D}b - b \cdot \mathcal{D}a. \quad (1.32)$$

The information encapsulated in this bracket structure is of utmost importance in relating the  $\bar{h}$  and  $\omega$  fields, as will be shown in techniques to follow.

The GTG equivalent of the Riemann tensor is a bivector-valued function of a bivector and position,

$$\mathcal{R}(a \wedge b, x) = L_a \omega(b) - L_b \omega(a) + \omega(a) \times \omega(b) - \omega(c) \quad (1.33)$$

with  $c$  given by 1.32. Examining how  $\mathcal{R}$  transforms under local displacements and rotations we find

$$\mathcal{R}(B, x) \mapsto \mathcal{R}(B, f(x)), \quad \mathcal{R}(B, x) \mapsto R\mathcal{R}(\tilde{R}BR, x)\tilde{R}, \quad (1.34)$$

respectively. We refer to functions with these transformation properties as covariant, and denote them by script-style letters. It is a simple matter to construct the typical contractions

$$\mathcal{R}(b, x) = \partial_a \mathcal{R}(a \wedge b, x), \quad (1.35)$$

$$\mathcal{R}(x) = \partial_b \mathcal{R}(b, x), \quad (1.36)$$

$$\mathcal{G}(a, x) = \mathcal{R}(a, x) - \frac{1}{2} \mathcal{R}(x)a, \quad (1.37)$$

the Ricci tensor, Ricci scalar and the Einstein tensor, respectively. To alleviate notational clutter, for these and other relevant functions we usually suppress the explicit dependence on  $x$ ; for example we write  $\mathcal{R}(a)$  rather than  $\mathcal{R}(a, x)$ .

Of significant interest in the work to follow is the Weyl tensor, defined by

$$\mathcal{W}(a \wedge b) = \mathcal{R}(a \wedge b) - 4\pi(a \wedge \mathcal{T}(b) + \mathcal{T}(a) \wedge b - \frac{2}{3}\mathcal{T}a \wedge b), \quad (1.38)$$

where  $\mathcal{T}(a)$  is the stress energy tensor and  $\mathcal{T}$  is its trace. We note that  $\mathcal{W}$  is self-dual, symmetric and trace-free

$$\mathcal{W}(IB) = I\mathcal{W}(B), \quad \sigma_\mu \mathcal{W}(\sigma_\mu) = 0. \quad (1.39)$$

It is important to note that in GTG we interpret these quantities as constructions on a flat background space rather than true curvature. However, often we will mix language and refer to spacetimes with vanishing  $\mathcal{R}(a \wedge b)$  as being flat.

We obtain field equations for  $\bar{h}$  and  $\Omega$  from the invariant action<sup>2</sup>

$$S = \int |d^4x| \det(\underline{h}^{-1})(\mathcal{R}/2 - 8\pi\mathcal{L}_m) \quad (1.40)$$

where  $\mathcal{L}_m$  is the matter Lagrangian in which the gauge fields are assumed to appear undifferentiated. Assuming the cosmological constant and spin-density to vanish, we obtain

$$\mathcal{G}(a) = 8\pi\mathcal{T}(a) \quad (1.41)$$

$$\mathcal{D} \wedge \bar{h}(a) = \bar{h}(\nabla \wedge a). \quad (1.42)$$

One may recover the metric of General Relativity via

$$g_{\mu\nu} = \underline{h}^{-1}(e_\mu) \cdot \underline{h}^{-1}(e_\nu). \quad (1.43)$$

Typically in GR one *begins* with a particular form for the metric and proceeds to calculate, eliminating many degrees of freedom at the outset. The intrinsic method takes the opposite tack, working with the displacement-gauge covariant fields  $\bar{h}$  and  $\omega$  while retaining the rotational freedom until a mathematically natural or physically relevant choice of gauge presents itself. As a consequence, one deals with a much larger set of equations; however, these equations are all first order rather than the notoriously difficult second order equations of traditional GR. In this sense, much of GTG (in the absence of spin) is similar in spirit to tetrad or NP formalisms [22, 31], with the notable distinction of the inclusion of GA in GTG.

### 1.3 Petrov Classification

As shown in [22], algebraic characterizations of the Weyl tensor determine important physical properties of the gravitational fields, particularly in the absence of matter. Below we review the formal Petrov classification of the Weyl tensor and its interpretation for gravitational fields. Each Petrov type has an associated canonical form for  $\mathcal{W}$ ; the gauge in which  $\mathcal{W}$  takes one of these will be referred

<sup>2</sup>We are employing geometrized units,  $G = c = \hbar = 1$ .

to as the canonical gauge. We will describe Petrov types and canonical structure using methods from GA as shown in [15]. This differs from the traditional  $Q$ -matrix formalism [26] in two respects. First, we do not artificially impose any complex structure on the space of bivectors; it seems that the traditional approach has confused the pseudoscalar,  $I$  for the complex scalar,  $i$ . Second GA describes linear functions in frame-free notation, thereby exposing the pertinent frame-free (and hence physically meaningful) properties of the Petrov type whereas matrices are inherently dependent on coordinates. For these reasons, it is clear that classification using GA is superior to the traditional formalism.

To begin, we take an orthonormal basis for the space of bivectors,

$$\{F_1, F_2, F_3, IF_1, IF_2, IF_3\}, \quad F_i^2 = 1, \quad F_i F_j = -F_j F_i, \quad i \neq j \quad (1.44)$$

and then utilize the general form for a symmetric, traceless, self-dual bivector-valued linear form acting on bivectors,

$$\mathcal{W}(B) = \sum_{j,k=1}^3 F_j B F_k \alpha_{jk} \quad (1.45)$$

where  $\alpha_{jk} = \alpha_{kj}$  is a symmetric scalar+pseudoscalar-entried matrix and the tracefree requirement is satisfied,

$$\sum_{j,k} F_j \cdot F_k \alpha_{jk} = 0. \quad (1.46)$$

For arbitrary  $\mathcal{W}$ , the “electric” part of  $\mathcal{W}$  refers to the scalar part of  $\alpha_{jk}$ , and the “magnetic” part is the pseudoscalar part.

Analysis of Petrov types employs the so-called method of algebraic forms explored in [15], as we will diligently follow below, but before analyzing specific cases we give a Penrose diagram of Petrov types, depicted in 1.1. Type I Weyl tensors are the most general; solutions of type II, III, D, N, or O are referred to as *algebraically special* spacetimes. As we will find, the GA approach makes the Penrose diagram far more apparent at the level of the canonical forms, whereas matrix analysis tends to obscure it.

### Types I, D and O

A field is type I if and only if there is a choice of  $F_i$  that diagonalizes  $\alpha_{jk}$  in equation 1.45. Then we may write,

$$\mathcal{W}(B) = \alpha_1 F_1 B F_1 + \alpha_2 F_2 B F_2 + \alpha_3 F_3 B F_3, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0, \quad (1.47)$$

which clearly has eigenbivectors  $F_1$ ,  $F_2$  and  $F_3$ . Since

$$\sum_{k=1}^3 F_k B F_k = -B, \quad (1.48)$$

we may simplify 1.47

$$\mathcal{W}(B) = \mu_0 (F_1 B F_1 + B/3) + \mu_1 (F_2 B F_2 + B/3), \quad (1.49)$$

where  $\mu_0 = 2\alpha_1 + \alpha_2$ ,  $\mu_1 = 2\alpha_2 + \alpha_1$ . This is the canonical form for Petrov type I Weyl tensors. It is readily apparent that there are several algebraic

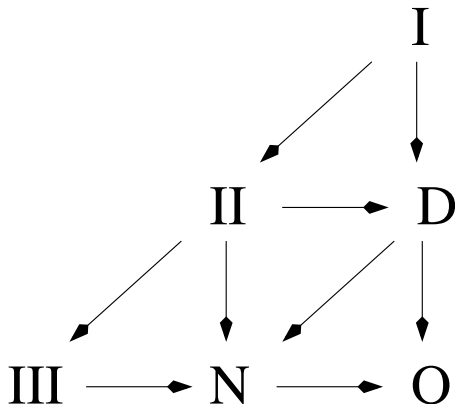


Figure 1.1: A Penrose diagram of Petrov types.

specializations of 1.49; of particular note is when one of the  $\mu_i$  vanishes, which may be shown to be equivalent to  $\mu_0 = \mu_1 = \mu$  using equation 1.48. We then find the type D canonical form,

$$\mathcal{W}(B) = \mu(F_1 B F_1 + B/3). \quad (1.50)$$

If both  $\mu_i$  vanish we find Petrov type O, which is simply

$$\mathcal{W}(B) = 0. \quad (1.51)$$

In a coordinate basis with bivectors  $\{\sigma_1, \sigma_2, \sigma_3\}$ , the type O canonical form is always achieved. For type D fields, we need only align  $F_1 = \sigma_1$ , which leaves freedom for rotations in  $I\sigma_1$  and boosts in  $\sigma_1$ . Achieving type I canonical form expends all rotational freedom and there is no invariance group.

In the canonical gauge, a type I field can be described by Coulombic and transverse-wave forces. Type D fields are purely Coulombic, with the most well-known examples being the Schwarzschild and Kerr vacuums. A field is type O if and only if it is conformally flat, and for such solutions curvature is due entirely to the Ricci tensor. In particular, type O vacuum fields are flat.

### Types II and N

A field is type II when  $\alpha_{jk}$  of equation 1.45 is diagonalizable with one of the  $F_i$  allowed to be null. We therefore define

$$F_0 = F_2 + IF_3, \quad \tilde{F}_1 = F_1 + \eta F_0 \quad \text{such that} \quad F_0 \cdot F_1 = 0 \quad (1.52)$$

where  $\eta$  is a scalar+pseudoscalar. The associated canonical form is

$$\mathcal{W}(B) = \mu_0 F_0 B F_0 + \mu_1 (\tilde{F}_1 B \tilde{F}_1 + B/3) \quad (1.53)$$

$$= (\mu_0 + \mu_1 \eta) F_0 B F_0 + \mu_1 (F_1 B F_1 + B/3) + \mu_1 \eta (F_1 B F_0 + F_0 B F_1), \quad (1.54)$$

where the second expression explicitly decomposes  $\mathcal{W}$  into type N, type D and type III components. We again find varying degrees of algebraic specialization;

taking  $\mu_0 = 0$  gives type D as above,  $\mu_0 = \mu_1 = 0$  is type O and  $\mu_1 = 0$  gives type N,

$$\mathcal{W}(B) = F_0 B F_0, \quad (1.55)$$

where we have rescaled the  $\gamma_0$  direction so that  $\mu_0 = 1$ . Note that  $\mathcal{W}^2(B) = 0$  and  $F_0$  is an eigenbivector. Taking a coordinate basis, for 1.53 we may choose  $F_2 = \sigma_2$ ,  $F_3 = \sigma_3$ , and  $\tilde{F}_1 = \sigma_1$ . Clearly, this expends all rotational freedom. On the other hand, for 1.55 we need only set  $F_2 = \sigma_2$ ,  $F_3 = \sigma_3$ , making the canonical type N form invariant under the null rotations that leave  $F_0$  fixed, or equivalently rotations about  $\gamma_0 + \gamma_1$ .

Type N fields describe transverse gravitational waves. Type II fields can be thought of as some combination of the physical circumstances leading to N, III and D types, which is a trivial observation given 1.54 but difficult to see in the  $Q$ -matrix formalism.

### Type III

Type III fields also have a null eigenbivector,  $F_0$ , and take the canonical form

$$\mathcal{W}(B) = \mu(F_0 B \tilde{F}_1 + \tilde{F}_1 B F_0), \quad (1.56)$$

with definitions for  $F_0$  and  $\tilde{F}_1$  as above. Notice that  $\mathcal{W}^3 = 0$ ,  $\mathcal{W}^2 \neq 0$ . Aside from type O, there are no algebraic specializations. When we choose a coordinate basis, all freedom is expended to set  $F_0 = \sigma_2 + I\sigma_3$ ,  $\tilde{F}_1 = \sigma_1$  and a rescaling of the  $\gamma_0$  axis to set  $\mu = 1$ . Physically, type III fields describe longitudinal gravitational waves.

## 1.4 Spacelike $G_2$ Symmetry: Cylindrical and Planar Fields

Below we discuss the class of fields we wish to address and some general features of our assumptions. For cylindrically symmetric fields as in [27], we begin with the coordinates

$$t = x \cdot \gamma_0 \quad r = \sqrt{-(x \wedge \sigma_3)^2} \quad (1.57)$$

$$\tan \phi = \frac{x \cdot \gamma_2}{x \cdot \gamma_1} \quad z = x \cdot \gamma_3. \quad (1.58)$$

The associated coordinate frame is

$$e_t = \gamma_0 \quad e_r = \cos \phi \gamma_1 + \sin \phi \gamma_2 \quad (1.59)$$

$$e_\phi = r(-\sin \phi \gamma_1 + \cos \phi \gamma_2) \quad e_z = \gamma_3, \quad (1.60)$$

with reciprocal frame  $\{e^t, e^r, e^\phi, e^z\}$ . We take an  $\bar{h}$ -function

$$\begin{aligned} \bar{h}(e^t) &= f_1 e^t + f_3 e^r + f_2 r e^\phi \\ \bar{h}(e^r) &= g_2 e^t + g_1 e^r + g_3 r e^\phi \\ \bar{h}(e^\phi) &= h_2 e^t + h_3 e^r + h_1 r e^\phi \\ \bar{h}(e^z) &= i_1 e^z, \end{aligned} \quad (1.61)$$

where above and hereafter, all functions are of  $t$  and  $r$  unless otherwise stated. This choice of  $\bar{h}$  is invariant under boosts in  $\sigma_r$  and  $\sigma_\varphi$  and rotations in  $I\sigma_z$ . Defining  $\varphi$  as being the unit vector in the  $\phi$  direction, the  $\omega$ -field associated to  $\bar{h}$  is

$$\begin{aligned}
\omega(e_t) &= -T\sigma_r + A\sigma_\varphi + (K + h_2)I\sigma_z \\
\omega(e_r) &= C\sigma_r + \bar{K}\sigma_\varphi + (D + h_3)I\sigma_z \\
\omega(\varphi) &= M\sigma_r + B\sigma_\varphi + (h_1 - G)I\sigma_z \\
\omega(e_z) &= FI\sigma_r + \bar{G}I\sigma_\varphi + E\sigma_z,
\end{aligned} \tag{1.62}$$

where we have arranged the  $h_i$  so that no  $\bar{h}$ -function components appear in the intrinsic field equations. If the orbit of the symmetry has the topology of a plane, we employ coordinates  $\{t, x, y, z\}$  and write

$$\begin{aligned}
\bar{h}(\gamma^0) &= f_1\gamma^0 + f_3\gamma^1 + f_2\gamma^2 \\
\bar{h}(\gamma^1) &= g_2\gamma^0 + g_1\gamma^1 + g_3\gamma^2 \\
\bar{h}(\gamma^2) &= h_2\gamma^0 + h_3\gamma^1 + h_1\gamma^2 \\
\bar{h}(\gamma^3) &= i_1\gamma^3,
\end{aligned} \tag{1.63}$$

and

$$\begin{aligned}
\omega(\gamma_0) &= -T\sigma_1 + A\sigma_2 + KI\sigma_3 \\
\omega(\gamma_1) &= C\sigma_1 + \bar{K}\sigma_2 + DI\sigma_3 \\
\omega(\gamma_2) &= M\sigma_1 + B\sigma_2 - GI\sigma_3 \\
\omega(\gamma_3) &= FI\sigma_1 + \bar{G}I\sigma_2 + E\sigma_3,
\end{aligned} \tag{1.64}$$

where all functions depend on  $t$  and  $x$ .

It is by design that the intrinsic field equations for both types of symmetry are the same, which is possible because away from the  $z$ -axis the fields are indistinguishable: both admit identical local symmetry groups, so differences may only arise from global considerations. Unless otherwise stated we complete all computations in cylindrical coordinates. However, if it turns out that the fields do not admit a cylindrical interpretation, i.e. they do not satisfy the elementary flatness criteria (see below), we are free to change to cartesian coordinates without affecting any of the intrinsic equations.

### 1.4.1 $G_2$ Symmetries

Our choice for  $\bar{h}$  does not encompass all two-dimensional symmetries; to be precise 1.61 and 1.63 describe fields admitting

- i. an abelian  $G_2$ ,
- ii. with hypersurface orthogonal Killing vectors fields (KVF's),
- iii. acting on spacelike 2-surfaces (hereafter  $S_2$ ).

The topology of these orbits may be  $\mathbb{R} \times \mathbb{R}$  (planar symmetry),  $\mathbb{R} \times \mathbb{S}^1$  (local cylindrical symmetry), or  $\mathbb{S}^1 \times \mathbb{S}^1$  (toroidal symmetry). For reasons of brevity, in what follows unless explicitly stated cylindrical symmetry refers to all the symmetries described above.

Fields with two-dimensional symmetry need not have hypersurface orthogonal Killing vectors. In fact, a Killing vector  $\xi$  is hypersurface orthogonal if and only if it is the gradient of a function, which clearly need not hold in general circumstances. However, a few useful theorems show this is not as stringent a restriction as one might imagine [30, 31]:

**Theorem 1** *Let  $\xi$  and  $\psi$  be commuting spacelike KVF's and suppose that they are orthogonally transitive, i.e. the 2-surfaces orthogonal to the  $G_2$  orbits are integrable. Then either both  $\xi$  and  $\psi$  are hypersurface orthogonal or neither is.*

**Theorem 2** *Let  $\xi$  and  $\psi$  be two commuting KVF's and suppose either  $\xi$  or  $\psi$  vanish at at least one point. Then for vacuum fields, the two planes orthogonal to  $\psi$  and  $\xi$  are integrable.*

Furthermore, there are no known cylindrically symmetric solutions whose Killing vectors are not orthogonally transitive, and it is thought that orthogonal transitivity may be a consequence of the field equations [26]. Thus, our assumption of hypersurface orthogonality of both KVF's reduces to hypersurface orthogonality of a single KVF. In particular, the hypotheses of theorem 2 are satisfied for all globally defined cylindrical vacuum fields since the angular Killing field vanishes on axis.

Spacetimes with a hypersurface orthogonal abelian  $G_2$  have diagonalizable line elements [30] and typically authors begin by exploiting this fact. As we will find, partial diagonalization can be mathematically useful, but (at least for now) we proceed in full generality in order to demonstrate the physical meaning of our gauge choices.

### Cylindrical Symmetry and Elementary Flatness

The topology of the group orbits alone does not make a global solution cylindrically symmetric. Simple topological arguments show that the Killing vector with closed ( $\mathbb{S}^1$ ) orbits must vanish along a spacelike curve, which is interpretable as the axis of symmetry. To ensure Lorentzian geometry along this axis there are additional so-called ‘‘elementary flatness’’ criteria that must be satisfied [26]. In GR these conditions specify the circumference of an orbit a distance  $\delta r$  from the symmetry axis as  $2\pi\delta r$  to first order. In practice, this definition can be difficult to implement and has caused a fair deal of confusion in the past [9, 16, 32]. In GTG the elementary flatness criteria are simply that the  $\bar{h}$  and  $\omega$  fields must be independent of  $\phi$  along the  $z$  axis. In our case this requires

$$\begin{aligned}
 g_1 &\rightarrow rh_1, & g_3 &\rightarrow -rh_3, \\
 f_2 &\rightarrow 0, & f_3 &\rightarrow 0, & g_2 &\rightarrow 0, & h_2 &\rightarrow 0, \\
 D &\rightarrow -h_3, & G &\rightarrow h_1, & B &\rightarrow C, & \bar{K} &\rightarrow -M, \\
 T &\rightarrow 0, & A &\rightarrow 0, & F &\rightarrow 0, & \bar{G} &\rightarrow 0
 \end{aligned} \tag{1.65}$$

as  $r \rightarrow 0$ . Solutions whose group orbits have topology  $\mathbb{R} \times \mathbb{S}^1$  but fail to satisfy 1.65 must either be matched onto elementary flat source solutions or have a singularity on axis.

## Symmetry Groups $G_r \supset G_2$

In what follows, the  $G_2$  required by our definition of  $\bar{h}$  is often found to be a subgroup of a larger symmetry group of spacetime, so here we wish to enumerate the possibilities. Groups  $G_3$  with orbits  $S_2$  give rise to spherical and plane-symmetric spacetimes and are necessarily of Petrov type D or O [26]; we will encounter these in Chapter 3. For a  $G_r$ ,  $r \geq 3$  acting on a  $S_3$ , we obtain the homogeneous subclass including Bianchi types  $I$ ,  $VI_0$ ,  $VI_h$ , and  $V$ , and open and flat Friedman-Robertson-Walter cosmologies [30]; we find some of these solutions in Chapters 2 and 3. If the symmetry group acts on timelike hypersurfaces,  $T_3$ , we recover the stationary cylindrically symmetric spacetimes analyzed in [27]. Since this case has been analyzed extensively, we typically refrain from exploring stationary solutions. Finally, if the symmetry group acts on null orbits,  $N_3$ , we find gravitational wave solutions as explored in Chapter 4.

## 1.5 The Bracket Structure

In GTG, the intrinsic derivatives and their associated commutators determine the geometry of spacetime.<sup>3</sup> When fixing the gauge, we work to make the geometric picture as clear and simple as possible, searching for preferred coordinate systems by picking out commuting brackets.

For our choice of  $\bar{h}$ , the associated  $L$ -derivatives are

$$\begin{aligned} L_t &= f_1 \partial_t + g_2 \partial_r + h_2 \partial_\varphi \\ L_r &= f_3 \partial_t + g_1 \partial_r + h_3 \partial_\varphi \\ L_\varphi &= f_2 \partial_t + g_3 \partial_r + h_1 \partial_\varphi \\ L_z &= i_1 \partial_z. \end{aligned}$$

Their commutators, the “bracket relations,” are

$$\begin{aligned} [L_r, L_t] &= TL_t + CL_r + (K + \bar{K})L_\varphi \\ [L_r, L_\varphi] &= (\bar{K} - M)L_t + DL_r - GL_\varphi \\ [L_t, L_\varphi] &= AL_t + (K - M)L_r - BL_\varphi \\ [L_r, L_z] &= -\bar{G}L_z \\ [L_t, L_z] &= -EL_z \\ [L_\varphi, L_z] &= FL_z. \end{aligned}$$

By applying the brackets to an arbitrary function of  $t, r$  and  $\phi$  and collecting terms of like derivatives, we find the following equations relating the  $\bar{h}$  and  $\omega$

---

<sup>3</sup>Since we always interpret the background as being flat, here geometry refers to geodesic paths, areas, volumes, etc. as determined by  $\bar{h}$ .

fields:

$$\begin{aligned}
L_r f_1 - L_t f_3 &= T f_1 + (K + \bar{K}) f_2 + C f_3 \\
L_r g_2 - L_t g_1 &= T g_2 + (K + \bar{K}) g_3 + C g_1 \\
L_r h_2 - L_t h_3 &= T h_2 + (K + \bar{K}) h_1 + C h_3 \\
L_r f_2 - L_\varphi f_3 &= -G f_2 + (\bar{K} - M) f_1 + D f_3 \\
L_r g_3 - L_\varphi g_1 &= -G g_3 + (\bar{K} - M) g_2 + D g_1 \\
L_r h_1 - L_\varphi h_3 &= -G h_1 + (\bar{K} - M) h_2 + D h_3 \\
L_t f_2 - L_\varphi f_1 &= A f_1 + (K - M) f_3 - B f_2 \\
L_t g_3 - L_\varphi g_2 &= A g_2 + (K - M) g_1 - B g_3 \\
L_t h_1 - L_\varphi h_2 &= A h_2 + (K - M) h_3 - B h_1 \\
L_t i_1 &= -E i_1 \\
L_r i_1 &= -\bar{G} i_1 \\
L_\varphi i_1 &= F i_1.
\end{aligned} \tag{1.66}$$

## 1.6 Curvature and the Matter Field

After we have chosen  $\bar{h}$ , the content of our physical assumptions is placed in restrictions on the Riemann, Weyl, and stress-energy tensors. Below we give the general forms for these tensors in the presence of a dust perfect fluid matter field.

The Riemann tensor is given by

$$\begin{aligned}
\mathcal{R}(\sigma_r) &= \alpha_1 \sigma_r + \beta_1 I \sigma_z + \gamma_1 \sigma_\varphi \\
\mathcal{R}(\sigma_z) &= \alpha_6 I \sigma_r + \beta_6 \sigma_z + \gamma_6 I \sigma_\varphi \\
\mathcal{R}(\sigma_\varphi) &= \alpha_2 \sigma_r + \beta_2 I \sigma_z + \gamma_2 \sigma_\varphi \\
\mathcal{R}(I \sigma_r) &= \alpha_4 I \sigma_r + \beta_4 \sigma_z + \gamma_4 I \sigma_\varphi \\
\mathcal{R}(I \sigma_z) &= \alpha_3 \sigma_r + \beta_3 I \sigma_z + \gamma_3 \sigma_\varphi \\
\mathcal{R}(I \sigma_\varphi) &= \alpha_5 I \sigma_r + \beta_5 \sigma_z + \gamma_5 I \sigma_\varphi,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1 &= -L_r T - L_t C + T^2 - K(M + \bar{K}) - C^2 - \bar{K}M + AD \\
\alpha_2 &= -L_t M - L_\varphi T - A(G + T) + C(K - M) - B(K + M) \\
\alpha_3 &= L_\varphi C - L_r M - G(M + \bar{K}) + T(M - \bar{K}) - D(B - C) \\
\alpha_4 &= -L_\varphi F + F^2 - EB + G\bar{G} \\
\alpha_5 &= L_r F + \bar{G}(D + F) + E\bar{K} \\
\alpha_6 &= -L_t F - E(A + F) - \bar{G}K \\
\beta_1 &= L_r K - L_t D + G(K + \bar{K}) - T(K - \bar{K}) + C(A - D) \\
\beta_2 &= L_t G + L_\varphi K + D(K - M) + B(G + T) + A(K + M) \\
\beta_3 &= L_r G + L_\varphi D + G^2 + D^2 - K(M - \bar{K}) + \bar{K}M - BC \\
\beta_4 &= -L_\varphi E + F(E - B) + \bar{G}M \\
\beta_5 &= L_r E - \bar{G}(C - E) + F\bar{K} \\
\beta_6 &= -L_t E - E^2 - AF - \bar{G}T \\
\gamma_1 &= L_r A - L_t \bar{K} - B(K + \bar{K}) + C(K - \bar{K}) - T(A - D) \\
\gamma_2 &= L_\varphi A - L_t B + A^2 - B^2 + KM - GT + \bar{K}(K - M) \\
\gamma_3 &= L_\varphi \bar{K} - L_r B + D(M + \bar{K}) - A(M - \bar{K}) - G(B - C) \\
\gamma_4 &= -L_\varphi \bar{G} - F(G - \bar{G}) + EM \\
\gamma_5 &= L_r \bar{G} + \bar{G}^2 - EC - DF \\
\gamma_6 &= -L_t \bar{G} - E(\bar{G} + T) + FK.
\end{aligned}$$

The Einstein tensor is

$$\begin{aligned}
\mathcal{G}(e_t) &= -(\alpha_4 + \beta_3 + \gamma_5)e_t + (\beta_2 - \gamma_6)e_r + (\alpha_6 - \beta_1)\varphi \\
\mathcal{G}(e_r) &= (\gamma_3 - \beta_5)e_t - (\alpha_4 + \beta_6 + \gamma_2)e_r + (\gamma_1 - \alpha_5)\varphi \\
\mathcal{G}(\varphi) &= (\beta_4 - \alpha_3)e_t + (\alpha_2 - \gamma_4)e_r - (\alpha_1 + \beta_6 + \gamma_5)\varphi \\
\mathcal{G}(e_z) &= -(\alpha_1 + \beta_3 + \gamma_2)e_z.
\end{aligned}$$

For a dust matter field, the stress-energy tensor takes the form

$$\mathcal{T}(a) = \rho a \cdot v v \quad (1.67)$$

where  $\rho$  is the energy density and  $v$  is the covariant fluid velocity,  $v^2 = 1$ . In what follows, we always fix the rotation gauge by setting  $v = e_t$ . The contracted Bianchi identities

$$\mathcal{D} \cdot (\rho v) = 0, \quad (1.68)$$

$$\rho(v \cdot \mathcal{D}) \wedge v = 0, \quad (1.69)$$

for this case are

$$A = T = 0, \quad L_t \rho = -(B + C + E)\rho, \quad (1.70)$$

where the first equation ensures that the dust particles follow geodesic paths and the second relates the energy density along fluid streamlines to the expansion of space. In the cases analyzed here, the contracted Bianchi identities along with the bracket structure imply the full Bianchi identities.

Using equation 1.38 for the Weyl tensor and 1.67 for the stress-energy, the general form of the Weyl tensor is

$$\mathcal{W}(\sigma_r) = \left(\alpha_1 - \frac{4\pi\rho}{3}\right)\sigma_r + \beta_1 I\sigma_z + \gamma_1 \sigma_\varphi \quad (1.71)$$

$$\mathcal{W}(\sigma_\varphi) = \alpha_2 \sigma_r + \beta_2 I\sigma_z + \left(\gamma_2 - \frac{4\pi\rho}{3}\right)\sigma_\varphi \quad (1.72)$$

$$\mathcal{W}(\sigma_z) = \alpha_6 I\sigma_r + \left(\beta_6 - \frac{4\pi\rho}{3}\right)\sigma_z + \gamma_6 I\sigma_\varphi \quad (1.73)$$

$$\mathcal{W}(I\sigma_r) = \left(\alpha_4 + \frac{8\pi\rho}{3}\right)I\sigma_r + \beta_4 \sigma_z + \gamma_4 I\sigma_\varphi \quad (1.74)$$

$$\mathcal{W}(I\sigma_\varphi) = \alpha_5 I\sigma_r + \beta_5 I\sigma_z + \left(\gamma_5 + \frac{8\pi\rho}{3}\right)I\sigma_\varphi \quad (1.75)$$

$$\mathcal{W}(I\sigma_z) = \alpha_3 \sigma_r + \left(\beta_3 + \frac{8\pi\rho}{3}\right)I\sigma_z + \gamma_3 \sigma_\varphi \quad (1.76)$$

where self-duality of  $\mathcal{W}$  yields

$$\alpha_1 - \alpha_4 = \beta_6 - \beta_3 = \gamma_2 - \gamma_5 = 4\pi\rho \quad (1.77)$$

and

$$\alpha_2 = \alpha_5 \quad \beta_2 = -\beta_5 \quad \gamma_3 = -\gamma_6 \quad (1.78)$$

$$\gamma_1 = \gamma_4 \quad \alpha_3 = -\alpha_6 \quad \beta_1 = -\beta_4, \quad (1.79)$$

while that the Weyl tensor is symmetric gives

$$\alpha_2 = \gamma_1 \quad \alpha_3 = -\beta_1 \quad \gamma_3 = -\beta_2 \quad (1.80)$$

and its trace-free property has

$$\alpha_1 + \gamma_2 + \beta_6 = 4\pi\rho. \quad (1.81)$$

Recall that the gauge freedom in  $\bar{h}$  allows arbitrary boosts in  $\sigma_\varphi$  and  $\sigma_r$  and rotations in  $I\sigma_z$ ; therefore we have freedom to perform arbitrary rotations on the basis  $\{\sigma_r, \sigma_\varphi, I\sigma_z\}$  to put  $\mathcal{W}$  in a preferred form. However, since we do not have complete gauge freedom in the space of bivectors, assuming a particular form for  $\mathcal{W}$  requires additional physical assumptions on the fields. Of particular interest is diagonalization; by the spectral theorem of linear algebra necessary and sufficient conditions are that  $\mathcal{W}$  is symmetric on  $\sigma_r$ ,  $\sigma_\varphi$  and  $I\sigma_z$  when viewed as a  $3 \times 3$  matrix. This occurs precisely when the magnetic part of  $\mathcal{W}$  vanishes.

### The Intrinsic Fluid Velocity

As discussed in [8], the rotor parametrizing the change in frame between diagonalized  $\mathcal{W}$  and diagonalized  $\mathcal{T}$  defines an intrinsic velocity to a system. To be more precise, if the velocity of the fluid in the frame of diagonalized  $\mathcal{W}$  is

$$v = \cosh \chi e_t + \sinh \chi a, \quad (1.82)$$

for some spacelike vector  $a$ , then there is an intrinsic velocity defined by

$$v = \tanh \chi. \quad (1.83)$$

For the systems analyzed here diagonalized  $\mathcal{W}$  has simultaneously diagonalized  $\mathcal{T}$ . From this we draw several conclusions. First, a cylindrically symmetric system has vanishing intrinsic fluid velocity if and only if the Weyl tensor is pure electric in the frame of the dust,  $v = e_t$ . Second, systems with nonvanishing fluid velocities require rotations in at least one of  $I\sigma_r$ ,  $I\sigma_\varphi$  or  $\sigma_z$  to diagonalize  $\mathcal{W}$ . Third, since the physical interpretation of diagonalized  $\mathcal{W}$  is that the field appears to only have ‘Coulombic’ and transverse radiation components [26], we see a parallel to electromagnetism: (localized) radiation requires accelerating masses. That we may make such observations without explicit calculation of solutions demonstrates one of the strengths of the intrinsic method.

### Fluid Shear and Vorticity

To impose the physical assumptions on matter fields, we recall that the vorticity bivector, shear tensor and expansion scalar for a covariant velocity  $v$  are given by

$$\varpi = \mathcal{D} \wedge v + (v \cdot \mathcal{D}v) \wedge v, \quad (1.84)$$

$$\begin{aligned} \sigma(a) &= \frac{1}{2}(H(a) \cdot \mathcal{D}v + H(e^\mu)(e_\mu \cdot \mathcal{D}v) \cdot a) \\ &\quad - \frac{1}{3}H(e^\mu) \cdot (e_\mu \cdot \mathcal{D}v)H(a) \end{aligned} \quad (1.85)$$

$$\Theta = \mathcal{D} \cdot v \quad (1.86)$$

respectively, where

$$H(a) = a - a \cdot vv \quad (1.87)$$

is the projection operator. In our case,

$$\varpi = -(M - \bar{K})i\sigma_z \quad (1.88)$$

$$\sigma(e_t) = 0 \quad (1.89)$$

$$\sigma(e_r) = \frac{1}{2}(M + \bar{K})\varphi + \frac{1}{3}(2C - B - E)e_r \quad (1.90)$$

$$\sigma(\varphi) = \frac{1}{2}(M + \bar{K})\varphi + \frac{1}{3}(2B - C - E)e_r \quad (1.91)$$

$$\sigma(e_z) = \frac{1}{3}(2E - B - C)e_z. \quad (1.92)$$

$$\Theta = B + C + E. \quad (1.93)$$

In virtue of the Bianchi identities, time-dependence requires  $\Theta \neq 0$ . This turns out to be a critical importance in Chapter 2.

## 1.7 Type III Fields and a Useful Gauge

It can be extremely useful to expend two degrees of rotational freedom to put the Weyl tensor in block diagonal form,

$$\alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = \beta_1 = \beta_4 = \gamma_1 = \gamma_4 = 0, \quad (1.94)$$

so that

$$\mathcal{W}(B) = \mu_0(\sigma_r B \sigma_r + B/3) + \mu_1(FBF + F^2 B/3), \quad (1.95)$$

where  $\mu_0$  and  $\mu_1$  are scalars and  $F \in \text{span}\{\sigma_\phi, \sigma_z, I\sigma_\phi, I\sigma_z\}$  satisfies one of

$$F^2 = 1 \implies \text{Type I}, \quad (1.96)$$

$$F^2 = 0 \implies \text{Type II}. \quad (1.97)$$

The existence of such a gauge can be shown directly, or by noting that since there always exists sufficient gauge freedom to diagonalize the  $\bar{h}$ -function, if we fix

$$\begin{aligned} \bar{h}(e^t) &= f_1 e^t + f_3 e^r \\ \bar{h}(e^r) &= g_2 e^t + g_1 e^r \\ \bar{h}(e^\phi) &= h_1 r e^\phi \\ \bar{h}(e^z) &= i_1 e^z, \end{aligned} \quad (1.98)$$

then

$$A = D = F = K = \bar{K} = M = 0, \quad (1.99)$$

which has the desired effect on  $\mathcal{W}$ . Furthermore in this gauge

1. vorticity vanishes in the  $e_t$  frame,
2. the shear tensor is diagonal in the  $e_t$  frame,
3. all but two of equations 1.66 decouple,
4. for non-vacuum spacetimes the dust velocity is  $v = \cosh \chi e_t + \sinh \chi e_r$ .

Item 3 makes for very simple relations between  $\bar{h}$  and  $\omega$  component functions. Item 4 allows us to fix the  $\sigma_r$  freedom by  $v = e_t$ , which in turn diagonalizes  $\mathcal{T}$  and allows us to apply items 1 and 2 to the fluid streamlines, which we summarize along with previous results in:

**Lemma 3** *Consider a time-dependent dust solution admitting an abelian  $G_2$  on  $S_2$  with hypersurface orthogonal Killing vectors. Then in the frame of the dust, vorticity vanishes, expansion is non-vanishing and the shear tensor is diagonal.*

In virtue of 1.95, we may also conclude that all fields are type I or II—no type III solutions exist. Furthermore, type II solutions will have a luminal intrinsic fluid velocity since the transformation diagonalizing  $\mathcal{W}$  requires an ultraboost. We conclude that physically reasonable dust solutions will be of Petrov type I. Furthermore, specialization to type D fields requires that all eigenvalues be real. That we may make such progress without explicitly finding solutions is a typical feature of the intrinsic method.

## Chapter 2

# Rigidly Rotating Dust: Shear Simplicity

If the shear tensor vanishes along fluid streamlines, the dust is said to be rigidly rotating. Cylindrically symmetric rigidly rotating dust fields have provided some of the most interesting and bizarre exact solutions in GR [4, 9, 12, 29], being the first solutions discovered to demonstrate one of the major pathologies of the theory: closed timelike curves (CTC). Although the physical situations leading to such features are extreme, the very existence of causality violation calls into question the foundations of any theory of physics. For example, as shown in [27] for the van Stockum dust solution the intrinsic fluid velocity must become luminal to allow CTC; although this is comforting, it does not rule out their existence entirely.

In this chapter, we examine time-dependent shear-free fields with cylindrical symmetry, and show time dependence disallows CTC from forming. Furthermore we find that the intrinsic velocity necessarily vanishes. More precisely, the assumption of rigid rotation demands the fields either be stationary or a Friedmann-Robertson-Walker (FRW) cosmology. While this is far cry from solving the problem of causality-violation in GR and GTG, it is encouraging that there exists no time-dependent counterpart to the the van Stockum solution.

### 2.1 The Case Split: A Highbrow Perspective

We begin by fixing the gauge as in 1.7. The contracted Bianchi identities with the shear-free requirement reveal

$$T = 0, \quad B = C = E \quad \implies \quad L_t \rho = -3B\rho. \quad (2.1)$$

Along with the bracket structure, 2.1 imply the full Bianchi identities. Recalling that time-dependent solutions correspond precisely to  $3B = \Theta \neq 0$ , we invoke an extremely powerful theorem [6]:

**Theorem 4** *Any dust solution satisfying  $\sigma = \varpi = 0$ ,  $\Theta \neq 0$  along dust-particle streamlines is a Friedmann-Robertson-Walker cosmology.*

From the work in [27], stationary solutions ( $B = 0$ ) produce the van Stockum solution. These therefore represent the only two possibilities for rigidly rotating

dust. Although this general result is of interest, we find it instructive to carry out the explicit calculation to show the van Stockum solution is unique and that the time-dependent case yields all FRW metrics (flat, open and closed).

### 2.1.1 Time Dependence and FRW

If we assume  $B \neq 0$ , we find

$$\beta_2 = -\beta_2 = L_r B = 0 \quad (2.2)$$

which makes  $\mathcal{W}$  diagonal. Furthermore, since the trace of  $\mathcal{W}$  finds

$$L_t B + B^2 = -\frac{4\pi\rho}{3} \quad (2.3)$$

and the bracket relations show

$$[L_r, L_t] B = 0, \quad (2.4)$$

applying  $L_r$  to equation 2.3 shows that has  $L_r \rho = 0$ , so the solution is homogeneous. Furthermore,

$$\alpha_1 = \beta_6 = \gamma_2, \quad (2.5)$$

so  $\mathcal{W}$  is type O. We fix the displacement gauge as

$$h_1 = \frac{1}{ar} \implies G = g_1/r, \quad B = f_1 \frac{\partial_t a}{a} + g_2/r, \quad (2.6)$$

where  $a(t)$  is an arbitrary function. By inspection, there exists sufficient gauge freedom to diagonalize  $\bar{h}$ ,

$$f_3 = g_2 = 0. \quad (2.7)$$

Since then all functions with vanishing  $L_r$  derivatives are in fact functions of  $t$  alone and  $L_r f_1 = 0$ , we fix the  $t$  parametrization by

$$f_1 = 1. \quad (2.8)$$

The remaining hurdle is to integrate for  $g_1$  and  $\bar{G}$ , which is simplified dramatically by noting their  $t$ -dependence must be equal:

$$L_t g_1 = -B g_1, \quad L_t \bar{G} = -B \bar{G}. \quad (2.9)$$

We conclude that  $g_1$  and  $\bar{G}$  differ by an  $r$  dependent function so the equations

$$L_r g_1 = g_1 \bar{G}/r, \quad L_r \bar{G} + \bar{G}^2 = g_1 \bar{G}/r \quad (2.10)$$

find

$$g_1 = \frac{\sqrt{kr^2 + 1}}{a}, \quad \bar{G}(r) = \frac{kr}{a\sqrt{1 + kr^2}}. \quad (2.11)$$

We identify  $a$  as the scale factor

$$L_t g_1 = -B g_1, \quad L_t h_1 = -B h_1, \quad L_t i_1 = -B i_1, \quad (2.12)$$

and with the intrinsic field equations,

$$L_t B - G\bar{G} = -4\pi\rho, \quad L_t B + B^2 = -\frac{4\pi\rho}{3}, \quad (2.13)$$

we reproduce the initial value equation

$$\left(\frac{\partial_t a}{a}\right)^2 = -\frac{k}{a^2} + \frac{8\pi}{3}\rho. \quad (2.14)$$

Putting this together finds

$$\begin{aligned} \bar{h}(e^t) &= e^t & \bar{h}(e^r) &= a^{-1}\sqrt{1+k^2}e^r \\ \bar{h}(e^\phi) &= a^{-1}e^\phi & \bar{h}(e^z) &= \frac{e^z}{a\sqrt{1+kr^2}}, \end{aligned}$$

and a change to spherical coordinates shows this is indeed the FRW solution, as expected.

## 2.2 Stationary Dust and the van Stockum Interior Solution

As noted above,  $B = 0$  corresponds to a gauge invariant statement: the fields are stationary. If we therefore free the gauge completely, beginning with the assumptions

$$T = B = C = E = 0, \quad M + \bar{K} = 0, \quad v = e_t, \quad (2.15)$$

we may fix the remaining rotational freedom in the  $I\sigma_z$  plane by requiring

$$K = M. \quad (2.16)$$

This finds

$$K^2 = 2\pi\rho, \quad L_t\rho = 0. \quad (2.17)$$

Subsequently,

$$\alpha_2 = \alpha_5 = \gamma_1 = \gamma_3 = \gamma_4 = \gamma_6 = \beta_2 = \beta_5 = 0, \quad (2.18)$$

and

$$KF = \bar{G}D = 0. \quad (2.19)$$

Note that in the vacuum or vorticity-free case,  $K = \rho = 0$ , and

$$\alpha_i = \beta_i = \gamma_i = 0, \quad i = 1, 2, 3 \quad \implies \quad \mathcal{R}(B) = 0. \quad (2.20)$$

Further, if  $\bar{G} = F = 0$ , the field equations require

$$\alpha_1 + \beta_6 + \gamma_5 = M^2 = 0 \quad \implies \quad \mathcal{R}(B) = 0. \quad (2.21)$$

In light of 2.20 and 2.21, we choose

$$F = D = 0 \quad (2.22)$$

to satisfy equations 2.19. We then find that

$$M = K, \quad L_t G = L_t \bar{G} = L_t K = 0, \quad L_\varphi M = L_\varphi \bar{G} = 0 \quad (2.23)$$

and arrive at the field equations

$$L_r(G + \bar{G}) + G^2 + G\bar{G} + \bar{G}^2 - 3K^2 = -8\pi\rho \quad (2.24)$$

$$L_r K + \bar{G}K = 0 \quad (2.25)$$

$$L_r G + G^2 - K^2 = 0 \quad (2.26)$$

$$L_r \bar{G} + \bar{G}^2 + K^2 = 0 \quad (2.27)$$

$$G\bar{G} + K^2 = 0 \quad (2.28)$$

with non-vanishing brackets

$$[L_r, L_\varphi] = -2KL_t - GL_\varphi \quad (2.29)$$

$$[L_r, L_z] = -\bar{G}L_z. \quad (2.30)$$

These are precisely the equations derived in [27], and therefore yield the van Stockum solution for stationary rigidly rotating dust.

## 2.3 Conclusions

We have shown that a cylindrically symmetric field with dust stress-energy and vanishing shear is either the stationary van Stockum solution or an FRW cosmological model. For stationary rigidly rotating dust (as shown in [27]) it is precisely the intrinsic velocity that characterizes the three gauge sectors of the van Stockum solution: a subluminal velocity begets sensible causal structure, but at luminal and superluminal velocities CTC exist at all times. Time-dependence has been shown to exhibit no such CTC and does not break into analogous gauge sectors since the intrinsic velocity necessarily vanishes;

$$\mathcal{W} = 0, \quad (2.31)$$

so the Weyl tensor is diagonalized in any gauge, but in particular the gauge where  $\mathcal{T}$  is diagonalized. If time-dependence forbids non-vanishing rigid rotation, we can never expect systems to asymptotically relax to stationary systems like van Stockum without breaking cylindrical symmetry, having a nonvanishing shear or having nonvanishing pressure. Although this does not forbid any gauge sector of the van Stockum solution—and in particular does not rule out CTC in cylindrically symmetric fields—it does give valuable insight into how such systems might be created.

We interpret the vanishing of the Weyl tensor this to mean that time-dependent rigidly rotating cylinders are *non-rotating*. In fact we claim this is a physically intuitive notion. Recall that time-dependence demands non-vanishing expansion. Thus, imagining a dust cylinder as a collection of infinitesimal cylindrical shells, expansion with rotation will cause one shell to slide against its neighbors. If we are to retain symmetry, the only way to avoid such circumstances is by setting the intrinsic velocity or expansion to zero. As we have shown, vanishing expansion creates stationary fields and vanishing rotation yields the FRW solution. We investigate the case of zero rotation in a more general context in the next chapter to further understand the link between time-dependent systems and the intrinsic velocity.

## Chapter 3

# Non-Rotating Dust

In this chapter we explore dust solutions with pure electric Weyl tensors. As stated in 1.6, such systems have vanishing intrinsic velocities and are therefore non-rotating. The converse is also true: if the intrinsic velocity vanishes,  $\mathcal{W}$  is pure electric. Thus, the solutions found below represent the entirety of time-dependent solutions with vanishing intrinsic velocity.

They also are “silent universes,” being (i) dust models with (ii) vanishing vorticity and (iii) purely electric Weyl tensors. Such spacetimes are silent because information exchange between nearby areas of spacetime is prohibited—there are no sound waves (pressure is zero) or gravitational waves (the magnetic part of  $\mathcal{W}$  vanishes). In general, silent universes need not have any symmetries and were originally studied in the hopes of understanding perturbations during the inflationary era [19]. However, the extent to which they mimic the real universe has been a topic of some debate. For discussion on the possible objections, see [20]. However, this does not concern our analysis and in particular does not restrict the applicability of these solutions to astrophysical phenomena.

We begin by showing that there exist precisely five cases to consider: one of Petrov type O, one of type I, and three of type D. As we will find, all associated solutions have symmetry groups larger than  $G_2$ , the type O case being FRW, the type I an anisotropic Bianchi I universe and the three type D being stationary, spatially homogeneous and spherically symmetric. This result agrees with that in [2], though our analysis is considerably more compact due primarily to our assumptions of symmetry. From this we conclude that fields with dust stress energy tensors whose maximal symmetry group is a  $G_2$  on  $S_2$  must have a non-vanishing intrinsic fluid velocity.

### 3.1 Assumptions and Gauge Fixing

We again choose the gauge 1.7 though refrain from fixing  $v = e_t$ . Setting the remaining magnetic part of  $\mathcal{W}$  to zero

$$\beta_2 = \beta_5 = \gamma_3 = \gamma_6 = 0, \tag{3.1}$$

both diagonalizes  $\mathcal{W}$ ,

$$\begin{aligned}\mathcal{W}(\sigma_r) &= (\alpha_1 - \frac{4\pi\rho}{3})\sigma_r, & \mathcal{W}(I\sigma_r) &= (\alpha_4 + \frac{8\pi}{3}\rho)I\sigma_r, \\ \mathcal{W}(\sigma_\varphi) &= (\gamma_2 - \frac{4\pi\rho}{3})\sigma_\varphi, & \mathcal{W}(I\sigma_\varphi) &= (\gamma_5 + \frac{8\pi}{3}\rho)I\sigma_\varphi, \\ \mathcal{W}(\sigma_z) &= (\beta_6 - \frac{4\pi\rho}{3})\sigma_3, & \mathcal{W}(I\sigma_z) &= (\beta_3 + \frac{8\pi}{3}\rho)I\sigma_3.\end{aligned}\quad (3.2)$$

and has  $v = e_t$  automatically as a consequence of the field equations. This allows us to retain  $\sigma_r$  rotational freedom.

Taking  $L$ -derivatives of the greek-letter equations and simplifying with judicious application of the bracket relations, we obtain

$$L_t\alpha_4 = E(\gamma_2 - \alpha_4) + B(\beta_6 - \alpha_4) \quad (3.3)$$

$$L_r\alpha_4 = G(\gamma_5 - \alpha_4) + \bar{G}(\beta_3 - \alpha_4) \quad (3.4)$$

$$L_t\beta_3 = C(\gamma_2 - \beta_3) + B(\alpha_1 - \beta_3) \quad (3.5)$$

$$L_r\beta_6 = \bar{G}(\alpha_1 - \beta_6) - T(\gamma_5 - \beta_6) \quad (3.6)$$

$$L_t\gamma_5 = E(\alpha_1 - \gamma_5) + C(\beta_6 - \gamma_5) \quad (3.7)$$

$$L_r\gamma_2 = G(\alpha_1 - \gamma_2) - T(\beta_3 - \gamma_2). \quad (3.8)$$

For vacuum fields we set

$$\alpha_1 = \alpha_4 = \alpha, \quad \beta_3 = \beta_6 = \beta, \quad \gamma_2 = \gamma_5 = \gamma. \quad (3.9)$$

Equations 3.3-3.8 exhibit a triality symmetry, as first noted in [27], whereby simultaneous cyclic permutations of the (ordered) sets of functions

$$\{\alpha, \beta, \gamma\}, \quad \{B, C, E\}, \quad \{\bar{G}, T, G\} \quad (3.10)$$

leave the equations invariant. This feature can be viewed as a type of spatial homogeneity in solution space, and demonstrates symmetry under the exchange of bivectors (or more suggestively, spatial vectors)  $\sigma_r, \sigma_z$  and  $\sigma_\varphi$ . If we take the planar symmetric form of  $\bar{h}$ , this condition is precisely that of a Bianchi I spacetime. We stress this does not require the final solution to be isotropic; instead, if we take a solution and exchange the spatial directions we obtain a new (possibly distinct) solution. However, we can break the triality symmetry by assuming type D fields.

The contracted Bianchi identities are

$$T = 0, \quad L_t\rho = -(B + C + E)\rho, \quad (3.11)$$

which along with 3.3-3.8 imply the full Bianchi identities. The intrinsic field equations are

$$L_r(G + \bar{G}) + G^2 + \bar{G}^2 + G\bar{G} - EB - BC - EC = -8\pi\rho \quad (3.12)$$

$$L_t(B + E) + B^2 + E^2 + EB - G\bar{G} = 0 \quad (3.13)$$

$$L_r\bar{G} - L_t(C + E) + \bar{G}^2 - C^2 - E^2 - EC = 0 \quad (3.14)$$

$$L_rG - L_t(C + B) + G^2 - C^2 - B^2 - BC = 0 \quad (3.15)$$

$$L_tG + BG = 0 \quad (3.16)$$

$$L_t\bar{G} + E\bar{G} = 0 \quad (3.17)$$

$$L_rB + G(B - C) = 0 \quad (3.18)$$

$$L_rE + \bar{G}(C - E) = 0. \quad (3.19)$$

### 3.2 Type I Solutions: Kasner and FRW

Using the  $\sigma_r$  rotational freedom and displacement gauge freedom, we diagonalize the  $\bar{h}$  function and require  $h_1$  and  $i_1$  to be separable by multiplication, finding

$$B = B(t), \quad E = E(t), \quad (3.20)$$

and in particular the  $L_r$  derivatives of these functions vanish. Notice that if any two of  $B$ ,  $C$  or  $E$  are equal, the fields are type D. Thus,

$$0 = \gamma_3 = G(C - B) \implies G = 0 \quad (3.21)$$

$$0 = \beta_5 = -\bar{G}(C - E) \implies \bar{G} = 0, \quad (3.22)$$

from which we deduce

$$L_r \mathcal{W} = 0 \implies L_r \rho = 0, \quad C = C(t). \quad (3.23)$$

It is convenient to fix the  $t$  and  $r$  parametrizations by

$$f_1 = 1, \quad \partial_r g_1 = 0. \quad (3.24)$$

The non-vanishing components of the Riemann tensor simplify to

$$\alpha_1 = -L_t C - C^2 \quad \alpha_4 = -EB \quad (3.25)$$

$$\beta_6 = -L_t E - E^2 \quad \beta_3 = -BC \quad (3.25)$$

$$\gamma_2 = -L_t B - B^2 \quad \gamma_5 = -EC, \quad (3.26)$$

while the remaining equations 1.66 are

$$L_t h_1 = -B h_1, \quad L_t g_1 = -C g_1, \quad L_t i_1 = -E i_1. \quad (3.27)$$

Since elementary flatness cannot be satisfied ( $g_1 \rightarrow r h_1$  as  $r \rightarrow 0$ ), as discussed in 1.4 we may freely switch to cartesian coordinates. It is clear that the entire system retains the triality symmetry, and in fact that the  $x$ ,  $y$  and  $z$  directions are interchangeable in the field equations, as promised. If  $\rho \neq 0$ , this yields the FRW metric with  $k = 0$ . When  $\rho = 0$  we find

$$C = \frac{p_1}{t}, \quad B = \frac{p_2}{t}, \quad E = \frac{p_3}{t}. \quad (3.28)$$

where

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1. \quad (3.29)$$

It is a trivial matter to find the  $\bar{h}$ -function components, yielding

$$\bar{h}(a) = a \cdot \gamma_0 \gamma^0 + t^{-p_1} a \cdot \gamma_1 \gamma^1 + t^{-p_2} a \cdot \gamma_2 \gamma^2 + t^{-p_3} a \cdot \gamma_3 \gamma^3, \quad (3.30)$$

which is the Kasner solution,

$$ds^2 = dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2. \quad (3.31)$$

This is a prototypical example of a homogeneous, non-isotropic cosmology: two directions expand with  $t$  and one contracts. It turns out to be of wider applicability than one might imagine, being used as a very close approximation to more complicated models of various epochs of the early universe [25, 30]. It has Weyl tensor

$$\mathcal{W}(B) = -\frac{1}{t^2} (p_2 p_3 \sigma_1 B \sigma_1 + p_1 p_3 \sigma_2 B \sigma_2 + p_1 p_2 \sigma_3 B \sigma_3), \quad (3.32)$$

so that the singularity at  $t = 0$  is indeed a true curvature singularity. For discussion of the Kasner solution, see [25, 30, 31].

### 3.3 Type D Solutions

There are three inequivalent type D cases to consider corresponding to

$$\mathcal{W}(B) = \alpha(\sigma_i B \sigma_i + B/3), \quad i = r, \varphi, z. \quad (3.33)$$

Tomasi [27] examined one ( $i = \varphi$ ) and found the existence of a timelike KVF so that the associated solution was stationary. We explore the two remaining cases below, both of which have three-dimensional symmetry groups. In one case, we find spatial homogeneity and in another spherical symmetry.

#### 3.3.1 Spatial Homogeneity: $\gamma_5 = \alpha_4$

Here the Weyl tensor is of the form

$$\mathcal{W}(B) = \frac{3\beta_6 - 4\pi\rho}{4}(\sigma_z B \sigma_z + B/3), \quad (3.34)$$

and the equations 3.3-3.8 show

$$\bar{G} = 0, \quad B = C. \quad (3.35)$$

Exploiting the Weyl symmetries, the field equations may then be written

$$L_t B + B^2 - EB = -4\pi\rho \quad (3.36)$$

$$L_t E + L_r G + E^2 + G^2 - B^2 = -4\pi\rho \quad (3.37)$$

$$2L_t B + 2B^2 + L_t E + E^2 = -4\pi\rho \quad (3.38)$$

$$L_t G + BG = 0 \quad (3.39)$$

$$L_r B = L_r E = 0, \quad (3.40)$$

from which we find

$$L_r \rho = L_r \mathcal{W} = 0. \quad (3.41)$$

We fix the rotation and displacement gauges by

$$f_1 = 1 \quad rh_1 = g_1, \quad \partial_r g_1 = 0, \quad g_2 = f_3 = 0, \quad (3.42)$$

which is possible because

$$L_t g_1 = L_t(rh_1) = -B g_1. \quad (3.43)$$

For vacuum solutions it is then a simple matter to integrate

$$\bar{h}(a) = a \cdot e_t e^t + t^{-2/3} a \cdot e_r e^r + t^{-2/3} a \cdot e_\phi e^\phi + t^{-1/3} a \cdot e_z e^z, \quad (3.44)$$

and

$$B = \frac{2}{3t}, \quad E = -\frac{1}{3t}, \quad G = \frac{1}{rt^{2/3}}, \quad (3.45)$$

which is the type D Kasner solution, now in (elementary flat) *cylindrical* coordinates. As before, the associated  $\rho \neq 0$  solution is the flat FRW.

### 3.3.2 Spherical Symmetry: $\beta_3 = \gamma_2$

In this final type D case,

$$\mathcal{W}(B) = \frac{3\alpha_1 - 4\pi\rho}{3}(\sigma_r B \sigma_r + B/3) \quad (3.46)$$

and from 3.3-3.8, we obtain

$$G = \bar{G}, \quad B = E. \quad (3.47)$$

The structure of the field equations is surprisingly similar to the type D spherical setup as found in [18], and indeed we find this to be the identical black hole solution.

We begin fixing the gauge by

$$h_1 = 1/r, \quad (3.48)$$

finding

$$B = g_2/r, \quad G = g_1/r, \quad (3.49)$$

We define an intrinsic mass,

$$M = -\frac{r^3}{2}\alpha_4, \quad (3.50)$$

with the properties

$$L_t M = 0 \quad (3.51)$$

$$L_r M = 4\pi r^2 g_1 \rho. \quad (3.52)$$

To make contact between  $M$  and a classical mass, we choose  $f_3 = 0$  so that

$$\partial_r M = 4\pi r^2 \rho. \quad (3.53)$$

This also allows us to solve for  $f_1$  as

$$L_r f_1 = T f_1 \implies f_1 = \epsilon(t) \exp\left(\int \frac{T}{g_1} dr\right). \quad (3.54)$$

To further fix the gauge, we set the  $t$  parametrization with  $\epsilon(t) = 1$ . If we make the change of variable,

$$\phi \mapsto \cos(\phi), \quad z \mapsto \theta, \quad (3.55)$$

we find the metric on  $I\sigma_r$  to be

$$-r^2 d\Omega^2 = -r^2(d\theta^2 + \sin^2\phi d\phi^2), \quad (3.56)$$

Indeed, the equations are now identical to those modeling Schwarzschild black holes in [18]. For the sake of completeness, notice that when  $\rho = 0$ ,  $M$  is constant and we regain a degree of rotation-gauge freedom, allowing

$$f_1 = 1 \implies T = 0. \quad (3.57)$$

We fix the  $r$  parametrization with

$$g_1 = 1, \quad (3.58)$$

and integrating the bracket structure we obtain

$$\bar{h}(a) = a - \sqrt{2M/ra} \cdot e_r e^t \quad (3.59)$$

and

$$C = -\frac{M}{g_2 r^2}, \quad (3.60)$$

which is the black hole solution of [18] in the Newtonian gauge. This solution and its associated analytical dust solution have been studied extensively in [18].

### 3.4 Type O Solutions

Type O solutions are defined by

$$\alpha_1 = \gamma_2 = \beta_6, \quad \alpha_4 = \gamma_5 = \beta_3, \quad (3.61)$$

and as is easily checked have

$$M = K = \bar{K} = 0, \quad B = C = E. \quad (3.62)$$

Therefore such fields admit a shear- and vorticity-free timelike geodesic congruence. As before, we conclude the solutions are FRW.

### 3.5 Conclusions and Future Work

We have shown that all dust solutions admitting  $G_2$  symmetry and having a vanishing intrinsic fluid velocity have symmetry groups  $G_r \supset G_2$ ,  $r \geq 3$ . In this way we see that a non-vanishing intrinsic fluid velocity is a crucial element to cylindrically symmetric systems with time-dependence. In virtue of comments in 1.7, we have further proved that no type D dust solutions admit a maximal  $G_2$ . Combining these results with those of Chapters 1 and 2, we find

**Theorem 5** *Dust solutions admitting a maximal hypersurface orthogonal abelian  $G_2$  have*

- i. a vanishing vorticity bivector, a non-vanishing diagonalized shear tensor and non-vanishing expansion scalar along streamlines of the dust particles,*
- ii. non-vanishing intrinsic fluid velocity and*
- iii. are type II when the intrinsic velocity is luminal and general type I otherwise.*

When applied to cosmologies, this result requires that inhomogeneous silent universes with vanishing cosmological constant admit a group strictly smaller than an abelian  $G_2$ , or have a  $G_2$  without hypersurface orthogonal Killing vectors.

Preliminary investigations have had substantial difficulty in producing a general solution meeting the requirements of theorem 5. Some useful simplifications include restricting the forms of the shear, expansion or intrinsic velocity in a physically meaningful and mathematical useful way. For example, suppose that  $R$  is the rotor parametrizing the change in frame between diagonalized  $\mathcal{T}$  and diagonalized  $\mathcal{W}$  (where we are assuming type I  $\mathcal{W}$ ). Then

$$R = \exp(\Phi\sigma_\varphi)\exp(\Psi\sigma_z) \quad (3.63)$$

and

$$R\mathcal{W}(\tilde{R}BR)\tilde{R} = \text{diag}(\alpha, \beta, \gamma). \quad (3.64)$$

Gauge-invariant restrictions on the intrinsic velocity may now be encapsulated by restrictions on  $R$ . For example,  $L_t R = 0$ ,  $L_r R = 0$ ,  $R = \text{constant}$ , or  $R$  has some relation to shear tensor or expansion, e.g.  $\Phi\Theta = \text{constant}$ . Such cases represent physically distinct systems which not only enhances intuition when approaching the problem, but also forms sensible case splits. Since it appears that the general solution is too complex to approach directly, the latter is a crucial ingredient in improved understanding of these fields.

In conclusion, the definition of an intrinsic velocity arises quite naturally in GTG as an important gauge invariant quantity, particularly in systems with axial symmetry. We have shown that it plays a critical role, hereto unnoticed, in time-dependent systems with 2-dimensional symmetry. From the gauge theory point of view this significance is expected: gauge invariant quantities are the only physically meaningful probes of the system in question and therefore require careful attention.

## Chapter 4

# Algebraically Special Fields

Diverging from our study of rotating dust, we now consider algebraically special vacuum and null dust solutions in the context of gravitational waves. We find a very natural path to a direct case split between the Robinson-Trautman solutions and Kundt's class for spacetimes with 2-dimensional symmetry and prove the Goldberg-Sachs theorem in this context. We find all plane wave solutions, and give examples of polarized, monochromatic, and impulsive waves. We discuss null dust solutions very briefly, showing some of the similarities and differences when compared to gravitational plane waves.

### 4.1 Fields of Type II, D, N and O

If we fix the gauge as in 1.7 and employ the analysis therein (specifically that no type III solutions exist), the most general algebraically special Weyl tensor is

$$\mathcal{W} = \frac{3\alpha}{4}(\sigma_r B \sigma_r + B/3) + \beta(\pm\sigma_\varphi + I\sigma_z)B(\pm\sigma_\varphi + I\sigma_z) \quad (4.1)$$

where

$$\beta_2 = -\beta_5 = -\gamma_3 = \gamma_6 = \beta, \quad \alpha_1 = \alpha_4 = \alpha \quad (4.2)$$

$$\beta_3 = \beta_6 = -\alpha/2 \mp \beta, \quad \gamma_2 = \gamma_5 = -\alpha/2 \pm \beta. \quad (4.3)$$

Note  $\mathcal{W}$  is type N when  $\alpha = 0$ , type D when  $\beta = 0$ , and type O (flat) when  $\alpha = \beta = 0$ . The plus-minus cases are related by time-reversal.

Taking  $L$ -derivatives of greek-letter equations and simplifying using the

bracket relations finds

$$L_t \alpha = -\frac{3}{2}(B + E)\alpha + \beta [(\bar{G} \pm E) - (G \pm B)] \quad (4.4)$$

$$L_r \alpha = -\frac{3}{2}(G + \bar{G})\alpha + \beta [(B \pm G) - (E \pm \bar{G})] \quad (4.5)$$

$$\frac{1}{2}L_t \alpha + L_{r \pm t} \beta = -\frac{3}{2}B\alpha + \beta [2(T \mp C) - (G \pm B)] \quad (4.6)$$

$$\frac{1}{2}L_t \alpha - L_{r \pm t} \beta = -\frac{3}{2}E\alpha - \beta [2(T \mp C) - (\bar{G} \pm E)] \quad (4.7)$$

$$\frac{1}{2}L_r \alpha + L_{t \pm r} \beta = -\frac{3}{2}\bar{G}\alpha - \beta [2(C \mp T) + E \pm \bar{G}] \quad (4.8)$$

$$\frac{1}{2}L_r \alpha - L_{t \pm r} \beta = -\frac{3}{2}G\alpha + \beta [2(C \mp T) + B \pm G]. \quad (4.9)$$

The above is equivalent to

$$(G \pm B) = (\bar{G} \pm E) \quad (4.10)$$

$$L_{r \pm t} \alpha = L_{r \pm t} \beta = 0. \quad (4.11)$$

Utilizing the former and

$$0 = \beta_3 - \gamma_5 \mp (\gamma_3 + \beta_5) = G^2 - \bar{G}^2 \pm (BG - \bar{G}E) \quad (4.12)$$

$$0 = \beta_2 + \gamma_6 \pm (\beta_6 - \gamma_2) = BG - \bar{G}E \pm (B^2 - E^2) \quad (4.13)$$

we find roots

$$\{E = B, G = \bar{G}\}, \quad \{E = \mp \bar{G}, B = \mp G\}. \quad (4.14)$$

By inspection, the former sets the magnetic part of  $\mathcal{W}$  to zero,

$$\beta_5 = -\gamma_3 = -\beta_5 = 0, \quad (4.15)$$

and is type D, yielding the Schwarzschild vacuum as we saw in 3.3.2. The latter has  $\alpha_4 = \alpha = 0$  and so is type N. For this case, it is useful to fix the  $\sigma_r$  rotational freedom so that  $e_t \pm e_r$  becomes a geodesic path,

$$C = \pm T. \quad (4.16)$$

Note further that  $e_t \pm e_r$  is a covariantly constant vector, and therefore a Killing vector. We readily find the field equations

$$L_u(G + \bar{G}) + G^2 + \bar{G}^2 + T(G + \bar{G}) = 0, \quad (4.17)$$

$$L_v G = L_v \bar{G} = L_v T = 0. \quad (4.18)$$

where

$$u = \frac{1}{\sqrt{2}}(t \mp r), \quad v = \frac{1}{\sqrt{2}}(t \pm r) \quad (4.19)$$

Clearly, there remains substantial freedom in the solutions. We will address some specific cases below, but first wish to discuss some optical properties of the solution.

### 4.1.1 An Aside: Geodesic Null Congruences

When studying gravitational waves in the geometric optics approximation, null congruences of curves give information about how radiation propagates through spacetime. There are typically two cases considered in the literature [26]:

- i. Kundt's class: non-expanding congruences modeling radiation from very distant objects, the classic example being plane-fronted polarized waves.
- ii. Robinson-Trautman solutions: expanding congruences, modeling radiation emanating from finite sources.

The interplay between symmetry, Petrov types and null congruences has been studied extensively and produced a wide array of powerful methods. For example [13],

**Theorem 6 (Goldberg-Sachs)** *A vacuum field is algebraically special if and only if it admits a shear-free geodesic null congruence of curves.*

Below we find expressions for the shear, vorticity, and expansion of geodesic null congruences in GTG and apply them to the above fields.

For a frame  $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ , a shear-free null geodesic congruence with tangent vector  $k = \gamma_0 + \gamma_1$  satisfies

$$k \cdot \mathcal{D}k = 0 \quad (4.20)$$

$$\gamma_2 \cdot (\gamma_2 \cdot \mathcal{D}k) - \gamma_3 \cdot (\gamma_3 \cdot \mathcal{D}k) = 0 \quad (4.21)$$

$$\gamma_2 \cdot (\gamma_3 \cdot \mathcal{D}k) + \gamma_3 \cdot (\gamma_2 \cdot \mathcal{D}k) = 0. \quad (4.22)$$

The vorticity of the congruence is

$$\varpi = \frac{1}{2} (\gamma_2 \cdot (\gamma_3 \cdot \mathcal{D}k) - \gamma_3 \cdot (\gamma_2 \cdot \mathcal{D}k)), \quad (4.23)$$

while the expansion is

$$\Theta = \frac{1}{2} (\gamma_2 \cdot (\gamma_2 \cdot \mathcal{D}k) + \gamma_3 \cdot (\gamma_3 \cdot \mathcal{D}k)). \quad (4.24)$$

Applying these formulae,  $k = e_t \pm e_r$  is geodesic and shear-free when

$$C = \pm T, \quad E \pm \bar{G} = B \pm G, \quad (4.25)$$

and we find

$$\Theta = -\frac{1}{2}((B \pm G) + (E \pm \bar{G})), \quad \varpi = 0. \quad (4.26)$$

In light of equation 4.10, our form for  $\mathcal{W}$  implies the hypothesis of the Goldberg-Sachs theorem. Considering the other direction, if the hypothesis is satisfied the vector  $k$  is covariantly constant so that equations 4.10-4.11 hold, showing that  $\mathcal{W}$  is necessarily algebraically special. In this way we have proved the theorem for spacetimes admitting a  $G_2$ .

With the congruence  $k = e_t \pm e_r$  being vorticity-free automatically, the roots 4.14 correspond to a direct case-split between Robinson-Trautman solutions and Kundt's class. The converse also holds: if we apply the restrictions  $\varpi = \Theta = 0$  or  $\varpi = 0, \Theta \neq 0$  to the shear-free congruence  $k$  we arrive at the same type N and type D fields, respectively. Physically, solutions to 4.17-4.17 may not represent finite sources. Indeed, with  $L_{t \pm r}$  is clearly a Killing vector [10]

**Theorem 7** *A non-flat vacuum field is a plane wave spacetime if and only if it admits an abelian  $G_3$  on a null 3-surface,  $N_3$ .*

Since an abelian  $G_3$  on  $N_3$  has an abelian subgroup  $G_2$  acting on  $S_2$ , we have produced all plane wave spacetimes within this class of solutions.<sup>1</sup> It is easy to see that in virtue of this third KVF, the fields necessarily violate elementary flatness. Therefore, we hereafter perform calculations in a cartesian frame, where we make the replacements,

$$r \mapsto x, \quad \phi \mapsto y, \quad (4.27)$$

and accordingly for  $u$  and  $v$ .

#### 4.1.2 Displacement Gauge Fixing and Properties of the General Solution

For the sake of clarity, we treat the case in which  $e_v = e_t - e_r$  is a KVF. Since

$$[L_u, L_v] = TL_v, \quad (4.28)$$

we hypothesize the existence of an integrating factor  $X$  with the properties

$$L_v X = 0, \quad L_u X = -TX, \quad (4.29)$$

which is easily confirmed to be consistent with the bracket structure,

$$[L_u, L_v] X = 0 \quad \implies \quad [XL_u, XL_v] = 0, \quad (4.30)$$

which allows us to fix the displacement gauge as

$$XL_t = \partial_t, \quad XL_x = \partial_x, \quad (4.31)$$

making it clear that  $X = 1/f_1 = 1/g_1$ . Once we have solved the field equation, it is a trivial matter to find the remaining  $\bar{h}$ -function components, via

$$\partial_u f_1 = T, \quad h_1 = \exp\left(-\int \frac{G}{f_1} du\right), \quad i_1 = \exp\left(-\int \frac{\bar{G}}{f_1} du\right). \quad (4.32)$$

We define the amplitude,  $A$ , and polarization,  $\Phi$ , of the wave by

$$\mathcal{W} = AR\mathcal{W}_N\tilde{R}, \quad (4.33)$$

$$R = \exp(\sigma_1\Phi), \quad (4.34)$$

where  $\mathcal{W}_N$  is the canonical type N Weyl tensor. Again, these definitions differ from the usual ones in that all values are real, and the rotor  $R$  is simply a spacetime rotor, not a complex rotation. As is easily seen, a polarization and amplitude uniquely define a solution. Important examples include when  $A$  is a pure sine function (monochromatic waves) and when the polarization is constant (linearly polarized waves). We will address these cases below.

<sup>1</sup>In fact, plane wave spacetimes have a  $G_5$  with an abelian  $G_3$  subgroup acting on an  $N_3$ . This is what makes the wavefronts truly planar.

### 4.1.3 Linearly Polarized Waves

The most conceptually simple gravitational wave solution is that for linearly polarized waves. Assuming

$$T = 0, \quad \implies \quad [L_x, L_t] = 0 \quad (4.35)$$

we may choose

$$f_1 = g_1 = 1. \quad (4.36)$$

Without loss of generality, define

$$h_1 = \frac{e^\beta}{L}, \quad i_1 = \frac{e^{-\beta}}{L} \quad (4.37)$$

where  $\beta$  and  $L$  are functions of  $u$  alone. Then the field equations reduce to

$$\partial_u^2 L + (\partial_u \beta)^2 L = 0, \quad (4.38)$$

which is a well-known solution [3, 10, 21] having line element

$$ds^2 = 2dudv - L^2(e^{-2\beta} dy^2 + e^{2\beta} dz^2) \quad (4.39)$$

and  $\bar{h}$  field

$$\bar{h}(a) = a - \frac{1}{L} \{ (L - e^\beta) a \cdot \gamma^2 \gamma^2 + (L - e^{-\beta}) a \cdot \gamma^3 \gamma^3 \} \quad (4.40)$$

It is a triviality to change the angle of polarization by simply applying the rotor 4.34; the amplitude of the wave is

$$A = \partial_u^2 \beta + (\partial_u \beta)^2 + \frac{\partial_u^2 L + 2\partial_u \beta \partial_u L}{L}. \quad (4.41)$$

As discussed in [21], when

$$L \sim 1, \quad \beta \text{ is "small"}, \quad (4.42)$$

we recover the results and interpretation of gravitational waves in the linear theory, wherein the forces of the wave are restricted to the  $I\sigma_x$  plane in which lengths compress in one direction and expand equally in another. With 4.42 having no gauge invariant meaning, its interpretation in gauge theory gravity is unclear. Yet  $\bar{h}$  is the identity on  $\sigma_x$ , restricting wave effects to  $I\sigma_x$ , and within that plane we see expansion in one direction and contraction in the other. The Weyl tensor shows these to be precisely equal and opposite. In this way analysis using the  $\bar{h}$  function and Weyl tensor provides substantially more understanding than the metric, allowing us to work in the full non-linear theory rather than forcing us to make ill-defined simplifications. We note, however, that this analysis can be mirrored in the NP formalism and therefore is not restricted to GTG.

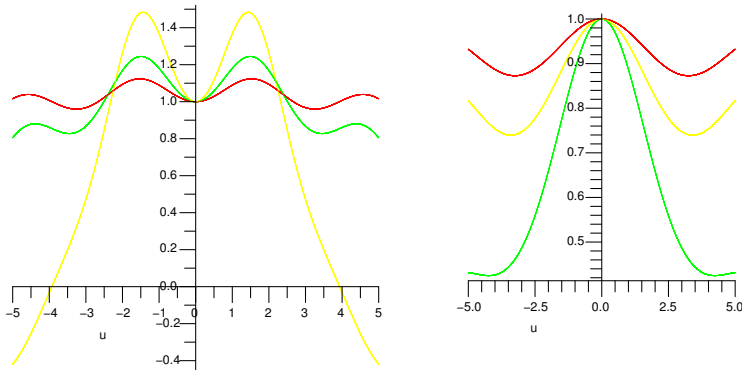


Figure 4.1: On the left are the Mathieu functions  $C(0, 1/8, u)$ ,  $C(0, 1/4, u)$  and  $C(0, 1/2, u)$  in red, green and yellow, respectively and on the right  $C(0, -1/8, u)$ ,  $C(0, -1/4, u)$  and  $C(0, -1/2, u)$ .

#### 4.1.4 Monochromatic Gravitational Plane Waves

Taking the most simple-minded type of wave oscillation, suppose

$$\mathcal{W} = A \sin(2ku) \mathcal{W}_N. \quad (4.43)$$

Since  $\Phi = 0$ , we take  $T = 0$  and find solutions

$$G = \partial_u \left( \frac{1}{C\left(0, \frac{A}{2k^2}, ku\right)} \right) \quad (4.44)$$

$$\bar{G} = \partial_u \left( \frac{1}{C\left(0, -\frac{A}{2k^2}, ku\right)} \right) \quad (4.45)$$

where  $C(n, q, u)$  is the even Mathieu function. This finds

$$h_1 = C\left(0, \frac{A}{2k^2}, ku\right), \quad i_1 = C\left(0, -\frac{A}{2k^2}, ku\right). \quad (4.46)$$

For small values of  $q$  near  $u = 0$ , the Mathieu functions appear periodic, though they are not globally so (when  $n = 0$ ). For convenience, we have included a graph of a few Mathieu functions in figure 4.1; note they have infinitely many zeros, so that the  $I\sigma_x$  plane collapses to a line due to the expansion in one direction and contraction in another associated with polarized gravitational waves. However, inspection of the Weyl tensor shows this is merely a coordinate singularity.

#### 4.1.5 An Impulsive Wave

First analyzed in [24], impulsive gravitational waves have flat spacetime everywhere except for a  $\delta$ -like curvature singularity along a null 2-surface. These waves have arisen naturally when considering observers moving very quickly

relative to Schwarzschild black holes [1]; more precisely, if the Schwarzschild coordinate system is ultraboosted (a spacelike direction becomes null) such that the energy of the black hole remains finite in the frame of the observer,  $M \rightarrow 0$  as  $v \rightarrow 1$ , an impulsive gravitational wave results. Thus, as an approximation to matter traveling at nearly the speed of light, impulsive gravitational waves are useful indeed. Collisions of impulsive waves have also been a subject of great interest [23]; surprisingly, spacelike singularities, horizons and black holes can be created in this way by backscattering [26]. The above systems have less symmetry than the  $G_3$  on  $N_3$  we impose here, so the solutions below are necessarily simpler. However, it is instructive to examine impulsive waves in this context.

In our case, consider the solution to 4.17,

$$T = 0, \quad G = -\frac{\theta(u)}{u-1}, \quad \bar{G} = -\frac{\theta(u)}{u+1} \quad (4.47)$$

where

$$\theta(u) = \begin{cases} 0 & \text{when } u < 0 \\ 1 & \text{when } u \geq 0. \end{cases} \quad (4.48)$$

Since

$$\partial_u G + G^2 = -\delta(u), \quad (4.49)$$

this gives precisely a type N impulse,

$$\mathcal{W} = \delta(u)\mathcal{W}_N, \quad (4.50)$$

with  $\bar{h}$ -field is

$$\bar{h}(a) = a + \theta(u)u(a \cdot \gamma^2 \gamma^2 - a \cdot \gamma^3 \gamma^3). \quad (4.51)$$

When compared to the metric describing impulsive waves, 4.51 is remarkably simple. In particular, it does not suffer from discontinuities or  $\delta$ -singularities. This makes the gauge theory approach not only more mathematically appealing, but also more physically reasonable since  $\delta$  discontinuities in the metric are difficult to interpret. By contrast, for the field 4.51 at  $u = 0$  there is a ‘‘kink’’ in the scale factor for the  $I\sigma_1$  plane, precisely of equal and opposite magnitude in the  $\gamma_2$  and  $\gamma_3$  directions. Clearly, before and after this change the fields have vanishing curvature and so the singularity at  $u = \pm 1$  is a coordinate singularity.

## 4.2 Some Comments on Null Dust

So-called null dust or pure radiation solutions model the gravitational field due to massless radiation. Possible physical sources include (classical) neutrinos or superpositions of electromagnetic plane waves with identical propagation directions but randomly oriented phases [26]. The associated stress energy tensor is

$$\mathcal{T}(a) = \rho a \cdot k k \quad (4.52)$$

where  $k^2 = 0$  is null. We fix the gauge as in 1.7, though now we remove the  $\sigma_1$  rotational freedom to set  $k = \gamma_0 - \gamma_1$ . The Bianchi identities will have  $k$

as a geodesic direction, and as discussed above, that direction is automatically shear-free. Assuming the fields to be algebraically special, we again find  $k$  to be a covariantly constant vector, and so we conclude the fields admit a group of isometries  $G_3$  on  $N_3$ . As one might expect, we find an extremely close relation between plane waves and null dust solutions.

To see this, if we pick arbitrary  $G$  and  $\bar{G}$  not necessarily solving the vacuum field equation 4.17 and *define* an energy density

$$L_r(G + \bar{G}) + G^2 + \bar{G}^2 + T(G + \bar{G}) = -8\pi\rho, \quad (4.53)$$

we obtain a solution with stress energy tensor

$$\mathcal{T}(a) = \rho a \cdot e_\nu e_\nu. \quad (4.54)$$

This relates to the following [26]:

**Theorem 8 (Mariot–Robinson)** *A spacetime admits a geodesic shear-free null congruence if and only if it admits an electromagnetic null field satisfying the Maxwell equations.*<sup>2</sup>

By allowing slightly more general  $\omega$  fields, null dust stress-energy can have more exotic solutions. The Weyl tensor takes the form

$$\mathcal{W}(\sigma_1) = 0 \quad (4.55)$$

$$\mathcal{W}(\sigma_2) = (\gamma_2 + 4\pi\rho)\sigma_2 + (\beta_2 - 4\pi\rho)I\sigma_3 \quad (4.56)$$

$$\mathcal{W}(\sigma_3) = (\beta_3 + 4\pi\rho)\sigma_3 + (\gamma_2 + 4\pi\rho)I\sigma_2 \quad (4.57)$$

$$\mathcal{W}(I\sigma_1) = 0 \quad (4.58)$$

$$\mathcal{W}(I\sigma_2) = (\gamma_5 + 4\pi\rho)I\sigma_2 + (\beta_5 - 4\pi\rho)\sigma_3 \quad (4.59)$$

$$\mathcal{W}(I\sigma_3) = (\beta_3 - 4\pi\rho)I\sigma_3 + (\gamma_3 - 4\pi\rho)\sigma_2, \quad (4.60)$$

which need not be type N. If we write

$$T = 0, \quad f_1 = g_1 = 1, \quad h_1 = e^{\beta_1}, \quad i_1 = e^{\beta_2}, \quad (4.61)$$

the field equation is simply

$$\partial_u^2(\beta_1 + \beta_2) + (\partial_u\beta_1)^2 + (\partial_u\beta_2)^2 = 8\pi\rho. \quad (4.62)$$

As a more concrete example, to produce an impulse of massless radiation we take

$$T = 0, \quad G = \bar{G} = \frac{\theta(u)}{u-1}, \quad (4.63)$$

finding an impulsive type N Weyl tensor with

$$\mathcal{T}(a) = \frac{\delta(u)}{4\pi} a \cdot e_\nu e_\nu, \quad (4.64)$$

and

$$\bar{h}(a) = a - \theta(u)u(a \cdot \gamma^2\gamma^2 + a \cdot \gamma^3\gamma^3). \quad (4.65)$$

<sup>2</sup>Every null electromagnetic field has a null dust interpretation, but the converse is not necessarily true.

Thus we see that null dust fields can produce very similar effects to gravitational waves.

As a point of contrast, for a solution with no vacuum counterpart take  $\mathcal{W} = 0$  which necessarily has a homogeneous distribution,  $\rho = \rho_0$ . We find the solution originally discovered by [5]

$$T = 0, \quad G = -\frac{k}{2} \frac{\cos(ku)}{1 - \sin(ku)}, \quad \bar{G} = \frac{k}{2} \frac{\cos(ku)}{1 + \sin(ku)} \quad (4.66)$$

and

$$h_1 = (1 - \sin(ku))^{-1/2}, \quad i_1 = (1 + \sin(ku))^{-1/2}. \quad (4.67)$$

where  $k = 4\sqrt{\pi\rho_0}$ .

### 4.3 Conclusions and Future Directions

We have provided all algebraically special vacuum solutions with two-dimensional symmetry; aside from solutions with pure electric  $\mathcal{W}$  explored in Chapter 3, this class is made up of plane wave spacetimes. We demonstrated that GTG provides an excellent language for interpreting gravitational wave solutions, with physical effects being far more readily apparent at the level of the  $\bar{h}$  function than the typical metric analysis. However, ultimately the Weyl tensor provides all the physically relevant information about the strength and orientation of the wave. It is the intrinsic method that allowed us to directly simplify and manipulate  $\mathcal{W}$ , producing a wide variety of solutions with immediate and clear physical interpretation. Using rotors, we have given a slightly different definition for polarization and amplitude without the usual dependence on complex numbers; although mathematically identical this constitutes a conceptual simplification. We have briefly examined the null dust counterparts to these solutions. Clearly there remains much work to be done in this arena, with possible specialization to Einstein-Maxwell fields.

In the process of finding these solutions, we have proved the Goldberg–Sachs theorem for spacetimes with an abelian  $G_2$ . It seems likely that a more general approach and proof could be produced within GTG, adapting and expanding methodology from the NP formalism. In addition, we expect the solutions we have found to be excellent starting points for more general studies of Robinson–Trautman solutions and Kundt’s class. Another avenue of possible future research is in the collision of plane waves. It seems that GTG with its flat background would be well-posed to provide a simple framework in which to produce well-defined matching conditions across the null hypersurfaces of collision.

### 4.4 Final Words

Throughout this work we have demonstrated GTG to be a particularly concise and descriptive language for gravitation. By focussing on gauge invariant quantities, namely the intrinsic velocity, we were able to make substantial progress in understanding cylindrically symmetric dust. By utilizing canonical Petrov forms, we enumerated all algebraically special vacuum solutions. We have demonstrated that the simple field equations produced by the intrinsic

method lead to much better physical and mathematical understanding of the resulting solutions. Geometric Algebra, and in particular its replacement of the complex scalar by the pseudoscalar, has kept the geometrical content of our equations clear providing obvious advantages over traditional Petrov classifications and the NP formalism.

In Chapter 1, after outlining GA, GTG and Petrov classification, we showed how the intrinsic method allows us to make general yet highly insightful observations about the fields in question. In particular, we showed that no type III solutions exist and that time-dependent dust has non-vanishing expansion and vanishing vorticity along fluid streamlines.

Chapter 2 continued the work of [27, 29] on rigidly rotating dust cylinders. We showed that the addition of time-dependence prohibits the pathological features of the stationary solutions, with time-dependent cylindrically-symmetric shear-free dust yielding the FRW solutions uniquely. Analysis of the Weyl tensor had as a trivial consequence that the dust was necessarily non-rotating, with the intrinsic velocity vanishing.

In Chapter 3 we continued to examine dust with vanishing intrinsic velocity, showing that such fields necessarily admit at least a  $G_3$  on either an  $S_2$  or  $S_3$ . Thus, we found that the rotation implied by the intrinsic velocity is a critical feature of fields admitting a maximal  $G_2$  on  $S_2$ . The combined results of Chapters 1, 2 and 3 allowed us to begin to piece together a general picture for time-dependent cylindrically-symmetric dust solutions.

In Chapter 4 we addressed topics somewhat removed from the first three chapters, beginning with algebraically special vacuum solutions. We found the Schwarzschild and gravitational plane wave solutions to be the only possibilities. Examining a few of the latter solutions in detail, GTG proved to be a very adept language over the traditional metric approach when describing gravitational waves. Very briefly we noted the connections between the algebraically special null dust solutions with 2-dimensional symmetry and gravitational plane waves.

In this work we have begun the task of probing the limitations imposed by 2-dimensional symmetry on gravitational fields. In this process, we have shown GTG to be a highly efficient and effective language for gravitation.

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