# THE CHERN-GAUSS-BONNET THEOREM VIA SUPERSYMMETRIC EUCLIDEAN FIELD THEORIES

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## 1. Introduction

In the last 30 years, supersymmetric field theory has inspired tremendous breakthroughs in topology and geometry. Starting with Witten's quantum mechanical description of Morse theory, on to Floer, Donaldson, Seiberg-Witten, and mirror symmetry, ideas from physics have provided motivation for wonderful mathematics. More recently, in the footsteps of Segal's work on conformal field theory, Stolz and Teichner have looked to $d|1$ supersymmetric Euclidean field theories as cocycles for generalized cohomology. The most notable—and indeed the most mysterious—is the cohomology theory of topological modular forms, TMF, which is conjectured to arise when $d = 2$. Below we apply methods from their program to field theories with more supersymmetry. Our motivation for this is twofold. First, mathematical understanding of quantization is more tractable in the presence of more supersymmetry, which allows one to gain new insight into some remaining open problems in the Stolz-Teichner program. Second, moduli spaces of supersymmetric field theories are interesting in their own right and can be used to extract various manifold invariants.

In addition to making steps towards the above goals, a primary objective of this paper is an alternate proof of the Chern-Gauss-Bonnet theorem wherein the Euler characteristic of a manifold is interpreted as the partition function of the $0|2$ sigma model. Subsequently, the Chern-Gauss-Bonnet formula can be seen as a supersymmetric localization for a *zero*-dimensional functional integral. In addition to this new perspective on the Euler characteristic, there is good reason to believe that such localization behavior persists for field theories with two supersymmetries, and so we believe this example to be a good one to understand thoroughly. The connection to cohomology in this story, though present, is not necessary for the Chern-Gauss-Bonnet theorem; it will be discussed in depth elsewhere [BE11]. Instead, out focus will be on how quantization in the presence of supersymmetry builds manifold invariants.

We would be remiss if we did not mention some of the previous field-theoretic proofs of the Chern-Gauss-Bonnet theorem. At a philosophical level all are motivated by considerations of supersymmetric quantum mechanics, which (as will be explained below) can be thought of as a $1|2$-Euclidean field theory. There is a smattering of mathematical literature emanating from physical observations due to Witten [Wit82] and Alvarez-Gaume [AG83];

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to take some examples, there are analytic—path integral or heat kernel—arguments in [Rog87, Lot87, BGV92, Roe98] and more algebraic ones—via the Mathai-Quillen formalism on loop space [MQ86]. At least at the formal level, all of these attempt to compute an integral over an infinite dimensional space and use various finite-dimensional localizations to confirm the Chern-Gauss-Bonnet formula.

The novelty of the proof in this paper has two main ingredients. First, all computations are centered around certain sheaves of $0|\delta$-Euclidean field theories on manifolds, which avoid all of the infinite dimensional considerations in the above-mentioned proofs while maintaining a field-theoretic interpretation. Second, we utilize ideas from physics and the Stolz-Teichner program [ST11] to show one can produce manifold invariants from these sheaves via quantization. The upshot of this approach is that—a priori—all partition functions of $0|\delta$-Euclidean field theories are manifold invariants. The proof of this is completely independent of the familiar supersymmetric cancelation arguments, thus providing new intuition for why supersymmetric field theories are so good at probing topological structures. We stress that constructing these quantizations boils down to writing down Gaussian measures in (finite-dimensional) supergeometry; there are no issues of functional integration. When $\delta = 2$, the computations are relatively easy, and some convenient deformations of the quantization map allow us to glean the Chern-Gauss-Bonnet formula.

At first, it seems to be a bit of a mystery that $0|2$ field theories compute anything interesting topologically; indeed the $0|1$ partition function invariant is totally boring. Dimensional reduction functors relate $0|2$ field theories to $1|2$-supersymmetric quantum mechanics, and this perspective lets us say a little more about the appearance of the Euler characteristic. We will describe this briefly at the end of the section.

## 1.1. $0|\delta$-Euclidean Field Theories.

We begin by giving a very brief introduction to supersymmetric Euclidean field theories. The full-blown definition is both lengthy (and somewhat incomplete, see [ST11]), but roughly follows in the footsteps of Atiyah, Kontsevich and Segal. Namely, a field theory is a symmetric monoidal functor[1] from the Euclidean $d|\delta$-bordism $d$-category over $X$ to some algebraic $d$-category, $\mathsf{ALG}$:

$$d|\delta\text{-}\mathsf{EFT}(X) := \mathsf{Fun}^{\otimes}_{\mathsf{SM}}(d|\delta\text{-}\mathsf{EB}(X), \mathsf{ALG}).$$

The algebraic target is the familiar category of vector spaces when $d = 1$, some delooping (or categorification) thereof for $d > 1$, and the looping (or decategorification) of $\mathsf{Vect}$ for $d = 0$, namely the commutative monoid $(\underline{\mathbb{R}}, \times)$, thought of as a symmetric monoidal 0-category. The definition of the $d$-category $d|\delta\text{-}\mathsf{EB}(X)$ is quite intricate in general. Naively, the $k$-morphisms form a generalized manifold of maps from $k|\delta$-Euclidean manifolds to $X$. This can be formalized in terms of an internal $d$-category in supermanifolds. Compositions are given by gluing these $k|\delta$-manifolds in a way that respects the Euclidean geometries. These details have yet to be understood completely, except in some lower dimensional examples [HKST09, HST09].

The approach of this paper is to stick to the *very* low dimensional theories: in dimensions $0|\delta$ these higher categorical complexities disappear, as $0|\delta$-bordisms are (roughly) a supermanifold of maps from $0|\delta$-manifolds to $X$. These toy models provide a nice mixture of richness and computability, allowing us to study and contrast features of field theories in the presence of varying amounts of supersymmetry.

To be more precise, in [HKST09] the authors show that the internal 0-category of $0|\delta$-Euclidean bordisms can be characterized as a certain quotient groupoid,

$$0|\delta\text{-}\mathsf{EB}(X) \simeq \underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)/\!/\,\underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta}),$$

where $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$ denotes the inner hom in the category $\mathsf{SM}$ of supermanifolds (to be explained in detail later) and $\underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta}) < \underline{\mathsf{Diff}}(\mathbb{R}^{0|\delta})$ is a chosen *Euclidean group* of isometries

[1] These are required to be fibered over supermanifolds, using the Grothendieck topology on supermanifolds of surjective submersions. We denote such fibered functors by $\mathsf{Fun}_{\mathsf{SM}}$. See [HKST09, ST11] for details.

of $\mathbb{R}^{0|\delta}$. With a little work, one can understand (fibered) functors to $\underline{\mathbb{R}}$ as functions,

$$\mathsf{Fun}^{\otimes}_{\mathsf{SM}}(0|\delta\text{-}\mathsf{EB}(X),\underline{\mathbb{R}})\cong C^\infty(0|\delta\text{-}\mathsf{EB}(X))$$

and we take this as our preliminary definition:

$$0|\delta\text{-}\mathsf{EFT}(X)\overset{\text{preliminary!}}{:=}C^\infty(0|\delta\text{-}\mathsf{EB}(X)).$$

We note that the inner hom is a representable supermanifold; if we choose a decomposition $\mathbb{R}^{0|\delta}\cong(\mathbb{R}^{0|1})^\delta$, we can iterate the isomorphism $\underline{\mathsf{SM}}(\mathbb{R}^{0|1},X)\cong\pi TX$, (which we will verify in Example 2.2.1) to obtain

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta},X)\cong(\pi T)^\delta X,$$

where here $\pi T\colon \mathsf{SM}\to\mathsf{SM}$ is the functor that takes a supermanifold to the total space of its odd tangent bundle, and $(\pi T)^\delta$ denotes $\delta$ applications of this functor. Thus, the quotient stack in our definition admits a description by a quotient groupoid in supermanifolds. We will now unpack this for some examples.

**Example 1.1** ($0|0\text{-}\mathsf{EFT}(X)$)**.** We have that $\mathbb{R}^{0|0}\cong\text{pt}$ as supermanifolds, and so

$$\underline{\mathsf{Euc}}(\mathbb{R}^{0|0}):=\{\text{id}\}\cong\underline{\mathsf{Diff}}(\mathbb{R}^{0|0})$$

Hence,

$$0|0\text{-}\mathsf{EFT}(X)\cong C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|0},X)/\!/\{\text{id}\})\cong C^\infty(X).$$

Given the simplicity of the 0-dimensional bordism category, one could have guessed this without all of these fancy definitions: the 0-bordism category is a 0-category whose objects are finite sets of points in $X$. Using the disjoint union (monoidal structure) this category is generated by single points in $X$. Thus, a symmetric monoidal functor to $\underline{\mathbb{R}}$ will be determined by its value on these. Putting this together, a 0-dimensional field theory assigns a number to each point in $X$, i.e., is a function on $X$. The reason for the seemingly complicated definitions is to ensure that this function be *smooth*; the language of fibered categories and fibered functors accomplishes this.

**Example 1.2** ($0|1\text{-}\mathsf{EFT}(X)$)**.** We choose the Euclidean group

$$\underline{\mathsf{Euc}}(\mathbb{R}^{0|1}):=\mathbb{R}^{0|1}\rtimes\mathbb{Z}/2<\underline{\mathsf{Diff}}(\mathbb{R}^{0|1})\cong\mathbb{R}^{0|1}\rtimes\mathbb{R}^\times,$$

and note that $C^\infty(\pi TX)\cong\Omega^\bullet(X)$, where differential forms are regarded as being $\mathbb{Z}/2$-graded via mod 2 reduction of the usual de Rham grading. We claim

$$0|1\text{-}\mathsf{EFT}(X)\cong C^\infty(\pi TX/\!/(\mathbb{R}^{0|1}\rtimes\mathbb{Z}/2))\cong\Omega^{\text{ev}}_{\text{cl}}(X),$$

where $\Omega^\bullet_{\text{cl}}$ denotes the sheaf of closed differential forms. One can see this by recalling that functions on the quotient are functions on $\pi TX$ invariant under the group action. Then a fun exercise (which we do in Example 2.3.1) shows that the infinitesimal action of $\mathbb{R}^{0|1}$ on $\Omega^\bullet(X)$ is precisely the de Rham $d$, and the $\mathbb{Z}/2$ action is by the grading involution. Hence, functions fixed under these two actions are $d$-closed and of even degree.

One might ask, can we obtain the odd forms via $0|1$-Euclidean field theories? The answer is affirmative, provided one uses so-called *twisted* field theories. Again, there is a general definition [HKST09], but we will be content to specialize to dimension $0|\delta$. A *twist* is a line bundle on $0|\delta\text{-}\mathsf{EB}(X)$, and a *twisted field theory* is a section of this line bundle. Note that a section of the trivial line bundle is just a function, which was our notion of an (untwisted) field theory above. Via a lemma in [HKST09], we make a (preliminary) definition

$$0|\delta\text{-}\mathsf{EFT}^{\mathcal{L}}(X)\overset{\text{preliminary!}}{:=}\Gamma(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta},X)/\!/\,\underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta}),\mathcal{L}).$$

One way to build a line bundle on $0|\delta\text{-}\mathsf{EB}(X)$ is to choose a homomorphism of super Lie groups,

$$\rho\colon\ \underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta})\to\mathbb{R}^\times.$$

We call the resulting line bundle $\mathcal{L}_\rho$. Its sections are functions on $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta},X)$ that are *equivariant* with respect to the induced action of $\underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta})$ on $\mathbb{R}$ via $\rho$. To see this, consider the more general situation of $M//G$ a quotient groupoid in $\mathsf{SM}$ and $\rho : G \to \mathbb{R}^\times$ is a homomorphism. Then (see Corollary 39 of [HKST09])

$$\Gamma(M//G, \mathcal{L}_\rho) \cong \{x \in C^\infty(M)^{\mathrm{ev}} \mid \mu^*(x) = p_1^*(x) \cdot p_2^*(\rho) \in C^\infty(M \times G)\},$$
$$\Gamma(M//G, \pi\mathcal{L}_\rho) \cong \{x \in C^\infty(M)^{\mathrm{odd}} \mid \mu^*(x) = p_1^*(x) \cdot p_2^*(\rho) \in C^\infty(M \times G)\}$$

where $p_1 \colon M \times G \to M$, $p_2 \colon M \times G \to G$ are the projections and $\mu : M \times G \to M$ is the action. Returning to field theories, notice that choosing $\rho$ to be the trivial homomorphism gives an ordinary (or untwisted) theory since equivariant functions with respect to the trivial action on $\mathbb{R}$ are exactly the invariant functions. The utility of the line bundles $\mathcal{L}_\rho$ is that they provide a graded ring $0|\delta\text{-}\mathsf{EFT}^{\mathcal{L}_\rho^{\otimes k}}(X)$ for each manifold $X$, where the multiplication comes from tensor products of line bundles. When the line bundle $\mathcal{L}_\rho$ is understood, we use the notation

$$0|\delta\text{-}\mathsf{EFT}^\bullet(X) := 0|\delta\text{-}\mathsf{EFT}^{\mathcal{L}_\rho^{\otimes\bullet}}(X).$$

**Example 1.3.** When $\delta = 1$, we choose the projection

$$\rho \colon \mathbb{R}^{0|1} \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2 \subset \mathbb{R}^\times,$$

to build a line bundle $\pi\mathcal{L}_\rho$. The functions equivariant with respect to the action of $\mathbb{R}^{0|1} \rtimes \mathbb{Z}/2$ are precisely the closed forms in the $-1$ eigenspace of the grading involution, i.e., the odd forms. Hence

$$0|1\text{-}\mathsf{EFT}^{\pi\mathcal{L}_\rho}(X) \cong \Omega^{\mathrm{odd}}_{\mathrm{cl}}(X).$$

The tensor product on line bundles gives Euclidean field theories twisted by $(\pi\mathcal{L}_\rho)^{\otimes k}$ for all $k$ the structure of a graded algebra. We've sketched the proof of the following result.

**Theorem 1.4** (Hohnhold-Kreck-Stolz-Teichner)**.** *There are isomorphisms of abelian groups*

$$0|1\text{-}\mathsf{EFT}^\bullet(X) \cong \begin{cases} \Omega^{\mathrm{ev}}_{\mathrm{cl}}(X) & \bullet = \text{even} \\ \Omega^{\mathrm{odd}}_{\mathrm{cl}}(X) & \bullet = \text{odd}. \end{cases}$$

*These isomorphisms are compatible with the graded ring structure on both sides, namely tensor products of field theories on the left and wedge products of forms on the right.*

1.2. **Concordance and Topological Invariants.** Given a closed differential form, one can extract a topological invariant by considering the de Rham cohomology class that form represents. Analogously, $0|\delta$-$\mathsf{EFT}$s give us new supergeometric objects generalizing closed forms, and similarly one can extract topological information from these. The correct notion that allows for this passage from the geometric to the topological is that of *concordance*, which we will explain presently.

First we note that a smooth map $f : X \to Y$ induces a functor $0|\delta\text{-}\mathsf{EB}(X) \to 0|\delta\text{-}\mathsf{EB}(Y)$, which in turn induces

$$f^* \colon 0|\delta\text{-}\mathsf{EFT}^{\mathcal{L}}(Y) \to 0|\delta\text{-}\mathsf{EFT}^{f^*\mathcal{L}}(X),$$

for $\mathcal{L}$ a line bundle on $0|\delta\text{-}\mathsf{EB}(Y)$.

**Definition 1.5.** Two field theories $\mathrm{E}_0, \mathrm{E}_1 \in 0|\delta\text{-}\mathsf{EFT}^{\mathcal{L}}(X)$ are concordant if there exists some $\mathrm{E} \in 0|\delta\text{-}\mathsf{EFT}^{p^*\mathcal{L}}(X \times \mathbb{R})$ such that

$$i_0^*\mathrm{E} = \mathrm{E}_0, \quad i_1^*\mathrm{E} = \mathrm{E}_1,$$

where $i_0, i_1 \colon X \hookrightarrow X \times \mathbb{R}$ are the inclusions $X \mapsto X \times \{0\}$ and $X \mapsto X \times \{1\}$, respectively, and $p \colon X \times \mathbb{R} \to X$ is the projection.

Concordance defines an equivalence relation on field theories, denoted $\sim$ throughout the paper. We use the notation

$$0|\delta\text{-}\mathsf{EFT}^\bullet[X] := 0|\delta\text{-}\mathsf{EFT}^\bullet(X)/\sim.$$

For field theories twisted by a line bundle $\mathcal{L}_\rho$, one can check that the graded ring structure on $0|\delta\text{-}\mathsf{EFT}^\bullet(X)$ descends to concordance classes. Thus, field theories furnish an additive homotopy functor from manifolds to graded rings,

$$0|\delta\text{-}\mathsf{EFT}^\bullet/\!\sim\, : \mathsf{Man}^{\mathsf{op}} \to \mathsf{GRing}.$$

In particular, the graded ring $0|\delta\text{-}\mathsf{EFT}^\bullet[X]$ is a topological invariant of $X$.

*Remark* 1.6. An application of Stokes theorem shows the following.

*Theorem* 1.7 (Hohnhold-Kreck-Stolz-Teichner). *There is an isomorphism of abelian groups*

$$0|1\text{-}\mathsf{EFT}^k[X] \cong \begin{cases} H^{\mathrm{ev}}_{\mathrm{dR}}(X) & k = \text{even} \\ H^{\mathrm{odd}}_{\mathrm{dR}}(X) & k = \text{odd} \end{cases}$$

*compatible with the ring structures on both sides, where $H^\bullet_{\mathrm{dR}}$ is de Rham cohomology with its usual cup product.*

In this paper we will focus on the simpler fact that concordance classes of $0|\delta$ field theories give additive homotopy functors.

1.3. **Sigma Models and Functional Integrals.** Sigma models are field theories that generalize ordinary mechanics. Recall that a starting point for classical mechanics is the generalized manifold of paths $\gamma\colon [0,t] \to X$ in a Riemannian manifold $(X, g)$; for simplicity we restrict our attention to loops,

$$\mathcal{F}_t X := \underline{\mathsf{SM}}(S^1_t, X).$$

The other essential ingredient in a classical theory is a function on $\mathcal{F}_t X$ called the *action*

$$\mathcal{S}(\gamma) := \int_{S^1_t} \left( \frac{1}{2} \|\dot{\gamma}\|^2 - \gamma^* h \right) dt,$$

that depends on some $h \in C^\infty(X)$, a *potential*, and a metric $g$ on $X$. This action is invariant under rotations of the circle, $S^1_t$, which we think of as a Euclidean isometry, $\underline{\mathsf{Euc}}(S^1_t)$.

The Wiener measure on loop space allows us to make sense out of the path integral quantization of this theory. We write

$$\langle \omega \rangle^{g,h}_t := \int_{\mathcal{F}_t X} \omega(\gamma) \frac{e^{-S(\gamma)}}{N(t)} \mathcal{D}\gamma,$$

where $\omega \in C^\infty(\mathcal{F}_t X)$ and the Wiener measure is

$$\mathcal{D}_t W = \frac{e^{-S(\gamma)}}{N(t)} \mathcal{D}\gamma.$$

Although $\mathcal{D}_t W$ has a rigorous definition, $\mathcal{D}\gamma$ and $N(t)$ only make sense heuristically: the former is some phantom "measure" and the latter is a (possibly infinite) normalization constant that depends on the dimension of $X$ and the circumference of the circle. Although there are a myriad of technical issues with the above expression, one can understand it rigorously via finite dimensional approximations to the Wiener measure; see [AD99, BP08] for discussions.

Sigma models attempt to generalize this story for the *space of fields*

$$\mathcal{F}_\sigma X := \underline{\mathsf{SM}}(\Sigma^{d|\delta}, X),$$

where $\sigma$ is some chosen geometry on $\Sigma$, generalizing the length of the interval in mechanics (again, for simplicity we are assuming that $\Sigma$ is closed). We equip the space of fields with an action functional similar to the one above: there is a kinetic term that computes something like the energy of a map, and a potential term that pulls back a function on $X$ and integrates it on $\Sigma$. Schematically,

$$\mathcal{S}(\Phi) := \int_\Sigma \left( \frac{1}{2} \|T\Phi\|^2 - \Phi^* h \right) \mathrm{vol}_{\Sigma,\sigma}, \quad \Phi \in \mathcal{F}_\sigma X.$$

As in the $1|0$-dimensional case above, $\|T\Phi\|^2$ depends on the metric on $X$, and this action is designed to be invariant under isometries of the worldsheet, $\underline{\mathsf{Euc}}(\Sigma)$.

Let $\mathcal{G}(X)$ denote the space of metrics and potential functions on the manifold $X$. We will call a choice of $g \in \mathcal{G}(X)$ a *geometry* for the sigma model on $X$.

| dim | partition function |
|---|---|
| $0\|1$ | 0, for $\dim(X) > 0$; signed cardinality when $\dim(X) = 0$ |
| $1\|1$ | $\hat{A}$-genus |
| $2\|1$ | Witten genus? |
| $0\|2$ | Euler characteristic |
| $1\|2$ | Euler characteristic and signature |

TABLE 1. In the listed dimensions, supersymmetric sigma models give (or are conjectured to give) the listed invariant. Conjectural statements are marked by "?".

**Definition 1.8** (Preliminary!)**.** We say that $d|\delta$-$\mathsf{EFT}$s have a (classical) sigma model if there is an action functional

$$\mathcal{S}(X,g) \in C^\infty(\underline{\mathsf{SM}}(\Sigma^{d|\delta}, X)/\!/\, \underline{\mathsf{Euc}}(\Sigma^{d|\delta})) \cong C^\infty(\underline{\mathsf{SM}}(\Sigma^{d|\delta}, X))^{\underline{\mathsf{Euc}}(\Sigma^{d|\delta})}$$

for any smooth manifold $X$ and choice of geometry $g \in \mathcal{G}(X)$. Furthermore, we require that this assignment is natural with respect to "isometric" immersions: if $i\colon (X,g) \to (Y,g')$ is an immersion, and $i^*g' = g$, we require that the induced map

$$C^\infty(\underline{\mathsf{SM}}(\Sigma^{d|\delta}, Y)) \to C^\infty(\underline{\mathsf{SM}}(\Sigma^{d|\delta}, X))$$

send $\mathcal{S}(Y,g') \mapsto \mathcal{S}(X,g)$. When the choices of target manifold and geometric data are clear, we denote $\mathcal{S}(X,g)$ by just $\mathcal{S}$.

*Remark* 1.9. This naturality condition is obeyed in examples, since sigma model actions are determined by local geometric data. There are, however, sigma models from physics that require more data (e.g., Kähler structures) than we have prescribed above, and hence would not fit in this definition. That said, one simply would need to tailor $\mathcal{G}$ to fit. Because of the local definition of the action in all examples, $\mathcal{G}$ will always be a sheaf.

In the dimensions where they exist, classical sigma models are easy to write down (see [Fre99]), but as we move into higher dimensions and add more geometric data (like supersymmetry) to $\Sigma$, we need to work hard to make sense out of the resulting functional integral quantization. Particularly in the presence of supersymmetry, there is good reason to believe that these integrals admit a rigorous definition and that they encode interesting topological data about $X$. The literature on this is voluminous, for example see [Wit82, DEF$^+$99, Fre99]. Here we wish to highlight the fact in all known examples, the *partition function*

$$Z_X(\sigma) := \langle 1\rangle_\sigma^g = \int_{\mathcal{F}_\sigma X} \frac{e^{-S(\Phi)}}{N}\mathcal{D}\Phi,$$

turns on to be an invariant of $X$, i.e., independent of the chosen geometry $g$. Some of the known examples are listed in table 1. One can show heuristically that $Z_X(\sigma)$ is additive and multiplicative on manifolds: additivity is automatic from axioms for field theories, and multiplicativity is "Fubini's theorem" for path integrals.

The goal of the next two subsections is formalize the above discussion on quantization in the case of $0|\delta$-$\mathsf{EFT}$s and give a proof that the associated partition functions are necessarily a manifold invariants.

1.4. **Quantization of $0|\delta$-$\mathsf{EFT}$s.** The above discussion gave some hints as to how supersymmetric field theories produce topological invariants; quantization is the key. Unfortunately, this is a very difficult notion to make mathematically rigorous. However, in dimensions $0|\delta$ the space of "fields,"

$$\mathcal{F}X := \underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X),$$

is a *finite* dimensional supermanifold, and (with some luck) we can make the above completely precise. Indeed, we will be able to mimic the physical intuition throughout, and when

$\delta = 1, 2$ the story is followed to the letter: there is an action function $\mathcal{S} \in C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$ leading to a quantization where $N$ is a finite normalization constant depending only on $\dim(X)$ and $\mathcal{D}\Phi$ can be defined via a Berezinian measure. See Appendix A for a brief review of integration on supermanifolds. We also note that the moduli of Euclidean geometries on $\mathbb{R}^{0|\delta}$ is a single point, so a partition function of a $0|\delta$ field theory will be a function on this point, so just a *number*.

**Example 1.10.** In dimension $0|1$ the situation is both easy and familiar. Here, the sigma model[2] is particularly simple: the fields are points of $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X) \cong \pi TX$, the action is $\mathcal{S} = 0$, and functions on fields are differential forms on $X$. The measure $\mathcal{D}\Phi$ projects to the top component of the given form and integrates on $X$. So for

$$\omega \in C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)) \cong C^\infty(\pi TX) \cong \Omega^\bullet(X),$$

we have

$$\langle \omega \rangle := \int_{\mathcal{F}X} \omega e^{-\mathcal{S}} \mathcal{D}\Phi = \int_{\pi TX} \omega = \int_X^{\mathrm{dR}} \omega,$$

where the last integral is the usual de Rham integral. Notice that the integral in this case is completely independent of geometric choices on $X$. Furthermore, we claim it preserves concordances: it suffices to show that null-concordant theories are mapped to null-concordant theories. But by Stokes theorem, $\omega \sim 0$ if and only if $\omega = d\alpha$ for some $\alpha$. By another application of Stokes,

$$\langle d\alpha \rangle = 0.$$

Also note that the partition function vanishes,

$$Z_X = \langle 1 \rangle = 0,$$

unless $X$ is a 0-manifold in which case $Z_X$ computes the signed cardinality of $X$. Although it is quite trivial, this is an additive, multiplicative topological invariant of oriented manifolds. The normalization constant, $N$, is set to 1 for this example.

*Remark* 1.11. We observe that to define the measure on the space of fields, no geometry on $X$ is required. However, $X$ must be oriented to carry out the integration, which (in physical language) is an example of an *anomaly*.

One might hope for a similar quantization map to exist for other Euclidean field theories, taking the form

$$\langle \omega \rangle^g := \int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)} \omega e^{-\mathcal{S}} \frac{\mathcal{D}\Phi}{N}, \tag{1}$$

where $\mathcal{D}\Phi$ is the Berezinian measure on $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$ described in Section 2.5. Even with this measure, we note that existence of a quantization is not automatic: the above integral might not converge, since $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$ is noncompact for $\delta > 1$. Indeed, this forces $\mathcal{S} \neq 0$ for $\delta > 1$ if a quantization is to exist—we require a Gaussian measure on fields. This prompts a definition.

**Definition 1.12.** Let $p\colon X \times Y \to Y$ be a projection whose fibers are of dimension $n$ and have the data of a metric and potential, and possibly an orientation. A *quantization* is a map of $0|\delta\text{-}\mathsf{EFT}^\bullet(Y)$-modules

$$\langle - \rangle^g \colon 0|\delta\text{-}\mathsf{EFT}^\bullet(X \times Y) \to 0|\delta\text{-}\mathsf{EFT}^{\bullet - n}(Y)$$

natural in $Y$ that has the form 1, for $\mathcal{S} \in C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$ and $N \in \mathbb{R}$.

[2] Physicists will probably disagree with the terminology; when we talk about the $0|\delta$ sigma model in this paper, we really mean the action functional of a dimensionally reduced sigma model.

1.5. **Partition Functions are Additive, Multiplicative Invariants.** Given a quantization, we'd like to get a manifold invariant. As explained above, concordance classes of field theories furnish invariants, so it will be critical to understand how quantization plays with concordance.

**Lemma 1.13.** *Quantization preserves concordance classes.*

*Proof.* Given field theories $E_0, E_1 \in 0|\delta\text{-}\mathsf{EFT}^\bullet(X \times Y)$ that are concordant via $E \in 0|\delta\text{-}\mathsf{EFT}^\bullet(X \times Y \times \mathbb{R})$, the quantization along the projection $p : X \times Y \times \mathbb{R} \to Y \times \mathbb{R}$ gives a concordance $\langle E\rangle^g \in 0|\delta\text{-}\mathsf{EFT}^\bullet(Y \times \mathbb{R})$ between $\langle E_0\rangle^g, \langle E_1\rangle^g \in 0|\delta\text{-}\mathsf{EFT}^\bullet(Y)$. □

The above lemma suggests that $\langle - \rangle^g$ could carry some topological meaning. However, a priori it seems to depend strongly on choices, namely the geometric data on $X$. Next we show that none of this extra data matter.

**Proposition 1.14.** *Assume that $0|\delta\text{-}\mathsf{EFT}$s have a sigma model, and let $\mathcal{S}_0$ and $\mathcal{S}_1$ be two action functionals depending on metrics $g_0, g_1$ on $X$ and potentials, $h_0, h_1 \in C^\infty X$. Then the actions are concordant, i.e., there exists an action functional on $X \times \mathbb{R}$ whose restriction to $0$ and $1$ is $\mathcal{S}_0$ and $\mathcal{S}_1$, respectively.*

*Proof.* We will construct these concordances explicitly. First, let $g_\lambda$ be a 1-parameter family of metrics connecting $g_0$ and $g_1$. Put the metric $\tilde{g} := g_\lambda \otimes d\lambda^2$ on $X \times \mathbb{R}$, where $\lambda$ is identified with a global coordinate on $\mathbb{R}$. Similarly, let $h_\lambda$ be a 1-parameter family of functions on $X$ connecting $h_0$ and $h_1$, which we can then promote to a function $\tilde{h}$ on $X \times \mathbb{R}$, $\tilde{h}(x, \lambda) = h_\lambda(x)$. Together, this gives an action functional $\mathcal{S}(X \times \mathbb{R}, \tilde{g}, \tilde{h})$. The inclusions of $X$ at 0 and 1 are isometric immersions, so by naturality of the sigma model the pulled back action functionals are the desired ones. □

**Corollary 1.15.** *Let $\langle - \rangle^g$ be a quantization. The induced map*

$$[\langle - \rangle^g] : 0|\delta\text{-}\mathsf{EFT}^\bullet[X \times Y] \to 0|\delta\text{-}\mathsf{EFT}^{\bullet - n}[Y],$$

*on concordance classes is independent of the choice of geometry $g$ on the fibers of $p : X \times Y \to Y$.*

*Proof.* The concordance of the Gaussian measures $e^{\mathcal{S}(g_\lambda, h_\lambda)}\mathcal{D}\Phi$ from the previous Lemma along with Lemma 1.13 will ensure that for a given $\omega$, the class determined by

$$\langle \omega \rangle^g = \int_{\mathcal{F}X} \omega e^{\mathcal{S}} \mathcal{D}\Phi$$

is independent of the choice of metric and potential. □

As mentioned above, our focus in this paper will be on invariants defined by partition functions. So given a quantization for $0|\delta\text{-}\mathsf{EFT}$s, we set $Y = \text{pt}$ and by Lemma 1.13 and Corollary 1.15, the quantity $\langle 1 \rangle^g$ defines a concordance class in $0|\delta\text{-}\mathsf{EFT}^{-n}(\text{pt})$ that is an invariant of $X$. To make further progress, we need a characterization of concordance classes of $0|\delta\text{-}\mathsf{EFT}^\bullet(\text{pt})$.

**Proposition 1.16.** *Field theories in $0|\delta\text{-}\mathsf{EFT}^\bullet(\text{pt})$ are concordant if and only if they are equal.*

**Corollary 1.17.** *Suppose that we have a quantization for $0|\delta\text{-}\mathsf{EFT}$s. Then the partition function*

$$Z_X = \langle 1 \rangle^g \in \mathbb{R}$$

*is independent of geometry $g$ on $X$, hence is an additive, multiplicative manifold invariant. In addition, the invariant is multiplicative under coverings.*

*Proof.* First we note that $0|\delta\text{-}\mathsf{EFT}^\bullet(\text{pt})$ are certain invariant elements of $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, \text{pt})) \cong \mathbb{R}$, so are just numbers. By Proposition 1.16, the concordance class of $\langle 1 \rangle^g \in \mathbb{R}$ is determined by the number $\langle 1 \rangle$, and by the above reasoning this number is independent of the metric and potential chosen on $X$. Additivity follows from basic properties of finite dimensional integrals and multiplicativity from Fubini's theorem.

To see the invariant is multiplicative under coverings, note that the integral computing the partition function invariant is local on $X$, so if we choose geometric data that is invariant under the permutation of the sheets of a covering $X \to Y$ (e.g., by averaging), the claim follows.□

Below we will show that in the case that $\delta = 2$ a map $\int_{\mathcal{F}}^{g}$ exists, and has all the desired properties above.

**Theorem 1.18.** *Let $p\colon X \times Y \to Y$ be a projection with metrics and potentials on the (n-dimensional) fibers. Via the Gaussian measure $e^{-\mathcal{S}}\frac{\mathcal{D}\Phi}{N}$ coming from the 0|2-sigma model, there is a quantization*

$$\langle - \rangle^{g} : 0|2\text{-}\mathsf{EFT}^{\bullet}(X \times Y) \to 0|2\text{-}\mathsf{EFT}^{\bullet - n}(Y). \tag{2}$$

*We emphasize that the fibers of $X \times Y \to Y$ need not be oriented.*

*Remark* 1.19. We expect aspects of the above story to generalize to $d|\delta$-$\mathsf{EFT}$s. After defining a particular notion of a $d|\delta$-$\mathsf{EFT}$, to obtain topological invariants following the above prescription our work is divided into two pieces: (1) construction of quantizations and (2) extraction of concordance invariants from $d|\delta$-$\mathsf{EFT}^{\bullet}(\mathrm{pt})$. The former is notoriously difficult. The latter can also be quite subtle, but fortunately we have many examples of deformation invariants of quantum field theories: indices of elliptic operators, chiral algebras, quantum cohomology, elliptic genera, etc.

For $0|\delta$ field theories, the moduli space of $0|\delta$ geometries is a single point so we get a partition number. However, for higher $d$ there may be many geometries on a closed connected $d|\delta$ manifold and $Z_X$ will be a function on this more interesting moduli space. For example in dimension 2|1—the relevant dimension for the Stolz-Teichner program—the Witten genus is the conjectured invariant, which can be thought of as a function (more precisely, section of a line bundle) on the moduli stack of elliptic curves.

Lastly, there are cases where the quantization may depend on more than a metric and potential wherein the resulting partition function invariant may not be topological; e.g., one might hope to find invariants of Kähler structures via the 0|4 sigma model. In general, the partition function will be an invariant depending on something like $\pi_0(\mathcal{G}(X))$ where $\mathcal{G}(X)$ is the space of geometries that allow one to define a quantization. However, the above definitions encompass all examples in this paper, and for the sake of simplicity we will restrict our attention to these.

1.6. **Localization and the Chern-Gauss-Bonnet Theorem.** We now turn our attention to the Chern-Gauss-Bonnet theorem, which will hinge on the definition of $\langle 1 \rangle^{g}$ when $\delta = 2$ and the resulting computation. Once defined, we know that $\langle 1 \rangle^{g}$ is an invariant of $X$. Furthermore, we already have a family of formulas equating invariants, for each pair of metric and potential. We will prove the Chern-Gauss-Bonnet theorem by identifying either side of the formula for some particular choices of metric and potential.

To better understand the action $\mathcal{S}$ and the Gaussian measure it defines, we need a geometric characterization of the mapping space.

**Lemma 1.20.** *Given a connection on $X$, there exists an isomorphism of supermanifolds*

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X) \cong p^*(\pi(TX \oplus TX))$$

*where $p\colon TX \to X$ is the usual projection.*

The purpose of the lemma is that after a choice of connection, we can think of a point in $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)$ as a point of $X$, two odd tangent vectors, and one even tangent vector; we denote this quadruple as $(x, \phi_1, \phi_2, F)$, respectively. This allows us to describe the action functional on $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)$ in terms of familiar geometric data on $X$.

**Lemma 1.21.** *Let $h \in C^\infty X$. The action functional for the* $0|2$ *sigma model with potential $h$ evaluated at a point $\Phi = (x, \phi_1, \phi_2, F) \in \underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)$ is*

$$\begin{aligned}\mathcal{S}(\Phi) &:= \int_{\mathbb{R}^{0|2}} (\frac{1}{2}\|T\Phi\|^2 - \Phi^* h)\mathrm{vol}_{\mathbb{R}^{0|2}} \\ &= \frac{1}{2}\|F\|^2/2 + \frac{1}{2}R(\phi_1, \phi_2, \phi_1, \phi_2)/2 - \langle F, \nabla h\rangle - \mathrm{Hess}(h)(\phi_1, \phi_2).\end{aligned}$$

We'll insert a formal parameter $\lambda \in \mathbb{R}$ in front of $h$, denoting the resulting 1-parameter family of action functions by $\mathcal{S}_\lambda$, and the pushforward with respect to this action by $\langle - \rangle^{g_\lambda}$. We'd like to calculate the partition function,

$$\langle 1 \rangle^{g_\lambda} = Z_X^\lambda = \int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)} e^{-\mathcal{S}_\lambda(\Phi)} \frac{\mathcal{D}\Phi}{N},$$

where we take $N = (2\pi)^{n/2}$, which (as advertised) only depends on the dimension of $X$. The value of this integral doesn't depend in an interesting way on $F$; physically this is because $F$ has algebraic equations of motion. Mathematically, this is because we integrate in the horizontal direction in the pullback diagram

$$\begin{array}{ccc} \underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X) & \to & \pi(TX \oplus TX) \\ \downarrow & & \downarrow \\ TX & \to & X \end{array}$$

which amounts to a Gaussian integral in the $F$-variable. The result is

$$Z_X^\lambda = \int_{\pi(TX \oplus TX)} \exp\left( -\frac{\lambda^2}{2} ||\nabla h||^2 + \lambda\, \mathrm{Hess}(h)(\phi_1, \phi_2) - R(\phi_1, \phi_2, \phi_1, \phi_2) \right).$$

If we set $\lambda = 0$, the odd integral will project $\exp(-R)$ onto the "top component," and we find that

$$Z_X^0 = \int_X \mathrm{Pf}(R),$$

so we have recovered one side of the Chern-Gauss-Bonnet formula. If we consider the limit $\lambda \to \infty$, we see that the integral for $Z_X^\infty$ will be supported on small neighborhoods of the set where $\nabla h = 0$. Assuming $h$ is Morse, there is a now-standard argument inspired by [Wit82] and due to [MQ86] showing

$$Z_X^\infty = (2\pi)^{n/2} \sum_{\nabla h = 0} \mathrm{sgn}(\nabla h) = (2\pi)^{n/2} \chi(X),$$

i.e., the integral computes the Hopf index of $\nabla h$.

Applying the fact that concordance in $0|\delta\text{-}\mathsf{EFT}^0(pt) \cong \mathbb{R}$ implies equality, we find

$$\int_X \mathrm{Pf}(R) = Z_X^0 = Z_X^\infty = (2\pi)^{n/2} \sum_{\nabla h = 0} \mathrm{sgn}(\nabla h),$$

and so have proved the Chern-Gauss-Bonnet theorem.

There is a physical interpretation of the above argument as a toy-model of a path integral localization: the sum over critical points is an integral over a formal neighborhood of the classical solutions of the sigma model action, whereas the integral of the Pfaffian comes from an integral over all fields. The former can be thought of as a 0-dimensional Feynman integral, whereas the latter is the "full" 0-dimensional path integral. This kind of localizing behavior in the quantization seems to persist for other Euclidean field theories with two supersymmetries; the $1|2$ case will be sketched presently.

1.7. **Relation to Supersymmetric Quantum Mechanics.** The discrete data of the above computation has a flavor of Morse theory to it, and this is no coincidence. In fact, one can generalize the entirety of the Chern-Gauss-Bonnet formula by considering $1|2$-supersymmetric quantum mechanics following [Wit82]. We wish to give a cartoon picture of this, strongly influenced by the discussion of $1|1$-$\mathsf{EFT}$s in [HST09].

Roughly, we have

$$1|2\text{-}\mathsf{EFT}(\mathrm{pt}) := \mathsf{Fun}^{\otimes}_{\mathsf{SM}}(1|2\text{-}\mathsf{EB}(\mathrm{pt}), \mathsf{Vect}).$$

On objects, the above functors assign a vector space to the super-point with two supersymmetries. This point has an automorphism group $O(2)$ gotten from rotating the odd directions, so $O(2)$ must act on the vector space. The $1|2$-intervals, which are certain morphisms in $1|2\text{-}\mathsf{EB}(\mathrm{pt})$, form a semigroup under composition (i.e., gluing of intervals), and a field theory will give a representation of this semigroup, which has infinitesimal generators:

$$Q_1, Q_2 \text{ odd}, \quad [Q_1, Q_2] = 0, \quad Q_1^2 = Q_2^2 = H. \tag{3}$$

There are some additional commutation relations with respect to the $O(2)$-action that we omit, and some pairings and copairings arising from the so-called left- and right-elbows in the bordism category. Rather than go into all the details here, we will give the most important example of a $1|2\text{-}\mathsf{EFT}(\mathrm{pt})$, the quantized $1|2$ sigma model of an even dimensional manifold.

We start with the vector space $\Omega^\bullet(X)$ for $X$ an even dimensional Riemannian manifold. We obtain an action of the Lie algebra of $\mathbb{R}^{1|2}_{>0}$ from odd generators $Q_1 = d + d^*$, $Q_2 = i(d - d^*)$ and $H = \Delta$, the Laplacian on forms on $X$. Note these operators satisfy 3. The action of $O(2)$ on $\Omega^\bullet(X)$ comes in two steps. First, we get an action of $S^1$ on $\Omega^\bullet(X)$ via the de Rham grading; to get an action of $O(2)$ we shift the grading so that the "middle" forms are of degree zero:

$$z \cdot \omega = z^{k-n/2}\omega, \quad \omega \in \Omega^k(X),\ z \in S^1.$$

This together with the $\mathbb{Z}/2$-action arising from the Hodge star (normalized so $\star^2 = 1$) gives a representation of $O(2)$ that intertwines appropriately with the $\mathbb{R}^{1|2}$-action. Finally, we assign the usual Hodge pairing

$$\langle \omega, \eta \rangle = \int_X \omega \wedge \star\eta$$

to the left elbow.

The supertraces of $e^{-t\Delta}$ with respect to the de Rham $\mathbb{Z}/2$-grading or Hodge $\mathbb{Z}/2$-grading give the Euler characteristic and signature of $X$, respectively. These are values of the $1|2$-partition function from evaluating the field theory on two different closed $1|2$-Euclidean manifolds. The Euler characteristic arises from evaluation on the circle with the trivial $\mathbb{R}^{0|2}$-bundle, and the signature comes from evaluation on a nontrivial $\mathbb{R}^{0|2}$-bundle.

We may employ a concordance similar to the one in the previous section, namely

$$d \mapsto d + \lambda dh$$

where $h \in C^\infty X$ and $\lambda \in \mathbb{R}$. This gives a 1-parameter family operators $Q_1^\lambda$ and $Q_2^\lambda$ acting on $\Omega^\bullet(X)$. One can check that for any $h$ and $\lambda$ the required commutation relations are satisfied so we obtain a concordance of $1|2\text{-}\mathsf{EFT}(\mathrm{pt})$. As shown (for example) in [Roe98, Zha11], in the limit $\lambda \to \infty$, the data of the field theory is precisely the Morse-Smale-Witten complex; the vector space is the free one on the critical points of $h$, and the differentials defining $Q_1$ and $Q_2$ arise from gradient flow-lines. The $O(2)$ grading is similar to the one in the case that $\lambda = 0$; the analogy of the Hodge star in the Morse case comes from "turning the manifold upside down."

We think of this 1-parameter family of field theories as defining a 1-parameter family of quantizations for field theories over $X$; roughly, a field theory over $X$ is a vector bundle $E$ with an $O(2)$-action, together with a pair of compatible connections $\tilde{Q}_1$ and $\tilde{Q}_2$, and quantization is

$$(E, \tilde{Q}_1, \tilde{Q}_2) \mapsto (\Gamma(E) \otimes \Omega^\bullet(X), \tilde{Q}_1 \otimes Q_1^\lambda, \tilde{Q}_2 \otimes Q_2^\lambda).$$

Thinking of this as arising from a path integral, we denote the quantization by $\int_{\mathcal{F}}^{g_\lambda}$, where $g_\lambda$ is the data of a (family of) metrics and Morse functions on $X$. Together with a dimensional reduction functor this makes a diagram:

$$\begin{array}{ccc} 1|2\text{-}\mathsf{EFT}(X) & \xrightarrow{\text{red}} & 0|2\text{-}\mathsf{EFT}(X) \\ {\scriptstyle \int_{\mathcal{F}}^{g_\lambda}}\big\downarrow & & \big\downarrow{\scriptstyle \langle - \rangle^g} \\ 1|2\text{-}\mathsf{EFT}(\text{pt}) & \xrightarrow{\text{red}} & 0|2\text{-}\mathsf{EFT}(\text{pt}), \end{array} \tag{4}$$

where for simplicity we have ignored the gradings. The horizontal arrows take the (super) trace of $e^{-tH}$, so for the quantized sigma model in $1|2$-$\mathsf{EFT}(\text{pt})$, this can be thought of as decategorification of the chain complex, $\Omega^\bullet(X)$, to the number, $\chi(X)$. The arguments above show that the diagram commutes for the unit in $1|2$-$\mathsf{EFT}(X)$, which is the trivial bundle with trivial connections—running in both directions, we get $\chi(X)$ in $0|2$-$\mathsf{EFT}(\text{pt})$. By varying the pushforwards, we get equivalences of many familiar descriptions of $\chi$: the alternating sum of Betti numbers in cohomology of $\Omega^\bullet(X)$; an alternating sum of critical points of $h$ a Morse function; the integral of the Pfaffian of the curvature; and the index of the vector field $\nabla h$. We should emphasize that the quantization for $1|2$-$\mathsf{EFT}$s is not yet on completely solid ground. However, the pushforward when $\lambda = \infty$ is surprising easy to define, and there are good notions of supersymmetric path integrals [Lot87, Rog87] for $\lambda$ finite.

Commutativity of the diagram 4 for the unit in $1|2$-$\mathsf{EFT}(X)$ is surprising. Indeed, the analogous statement for $1|1$-$\mathsf{EFT}$s is false. Recall that $1|1$-$\mathsf{EFT}(\text{pt})$ is a spectrum for $K$-theory, and one expects $1|1$-$\mathsf{EFT}(X)$ to give cocycles for $K^\bullet(X)$. Furthermore, there are diagrams

$$\begin{array}{ccc} 1|1\text{-}\mathsf{EFT}(X) & \stackrel{\text{red}}{\to} & 0|1\text{-}\mathsf{EFT}(X) \\ \int_{\mathcal{F}}\downarrow & & \downarrow\int \\ 1|1\text{-}\mathsf{EFT}(X) & \stackrel{\text{red}}{\to} & 0|1\text{-}\mathsf{EFT}(X) \end{array} \qquad \begin{array}{ccc} K^0(X) & \stackrel{ch}{\to} & H^{\text{ev}}_{\text{dR}}(X) \\ \pi_!\downarrow & & \int\downarrow \\ K^{-n}(\text{pt}) & \stackrel{ch}{\to} & H^{\text{ev}}_{\text{dR}}(X) \end{array}.$$

Quantization of a supervector bundle with super connection $(E, \nabla)$ on a spin manifold $X$ is

$$(E, \nabla) \mapsto (\Gamma(E) \otimes \Gamma(\$(X)), \nabla \otimes D)$$

where $\$(X)$ is the bundle on $X$ and $D$ is the Dirac operator. The content of the Atiyah-Singer theorem is that the diagram on the right commutes up to a factor of $\hat{A}(X)$. It seems very likely that the diagram on the left has a similar failure to commute. We note that as in the $1|2$ case, quantizations of $1|1$ field theories have yet to be defined rigorously.

With the above examples from $1|1$ and $1|2$ theories in mind, it is a somewhat puzzling fact that one can compute $\chi(X)$ completely within the context of $0|2$-$\mathsf{EFT}$s and quantization. Perhaps one explanation is that the Morse flow lines (or *instantons*) are unnecessary in the computation of $\chi$. Said differently, the information used in the the $1|2$-$\mathsf{EFT}$ quantization to compute $\chi(X)$ is contained in the critical points of $h$—the constant-path classical solutions—and hence is detected in the $0|2$-$\mathsf{EFT}$ quantization since the image of a $1|2$-$\mathsf{EFT}(X)$ under reduction to $0|2$ considers only the constant maps to $X$. This reveals features that appear to be unique to field theories with more than one supersymmetry. The practical upshot is that $0|2$-$\mathsf{EFT}$s allow us to avoid heat kernels and path integrals in the proof of the Chern-Gauss-Bonnet theorem, while still adhering to an interpretation from supersymmetric quantum mechanics.

1.8. **Notation and Conventions.** Before proceeding we need to pin down a little notation. We write $\mathsf{SM}$ for the category of supermanifolds, and refer the reader to [DEF$^+$99, HST11] for preliminaries. To be very brief, objects in this category are locally ringed spaces, $X^{n|m} = (|X|^n, C^\infty)$, where $C^\infty$ is a sheaf of superalgebras locally isomorphic to $C^\infty(\mathbb{R}^n) \otimes \Lambda^\bullet(\mathbb{R}^m)^*$. We write $|X|$ for the $n$-manifold $(|X|, C^\infty/\text{nilpotents})$, called the *reduced manifold* of $X$. Since smooth manifolds admit partitions of unity, morphisms of supermanifolds are determined by the induced map on global sections of the sheaf $C^\infty$, whence the slogan "supermanifolds are affine." We will use this fact without comment throughout.

Let $\mathsf{SM}(X,Y)$ denote the *set* of maps between supermanifolds $X$ and $Y$, and $\underline{\mathsf{SM}}(X,Y)$ for the inner hom, i.e., the functor

$$\underline{\mathsf{SM}}(X,Y)\colon \mathsf{SM}^{op}\to \mathsf{SET},\quad S\mapsto \mathsf{SM}(S\times X,Y).$$

Similarly, we denote $\underline{\mathrm{Diff}}(X)$ as the functor

$$\underline{\mathrm{Diff}}(X)(S)=\left\{\begin{array}{ccc} S\times X & \overset{\cong}{\to} & S\times X\\ & \searrow \quad \# \quad \swarrow & \\ & S & \end{array}\right\}.$$

The above are examples of *functors of points*, or for emphasis, *functors of $S$-points*. These functors may not be representable as supermanifolds, i.e., there may not exist a natural isomorphism with a functor

$$\underline{M}:\mathsf{SM}^{op}\to\mathsf{SET},\quad S\mapsto \mathsf{SM}(S,M),$$

where $M$ is some supermanifold. Still, much of supermanifold theory utilizes the functor of points rather than the supermanifold itself, and a surprising amount can be done with non-representable supermanifolds (so-called *generalized supermanifolds*). For example, the geometry of the free super loop space

$$\underline{sLX}(S):=\mathsf{SM}(S\times S^{1|1},X),\quad S^{1|1}:=S^1\times\mathbb{R}^{0|1}$$

is studied in [DEF$^+$99] in the context of mechanics. Defining (smooth) action functionals, observables, and classical solutions only requires the functor of points.

Even when a supermanifold is representable, whenever we refer to a point $\Phi$ of $M$, we implicitly mean a map $\Phi:S\to M$, so $\Phi\in\underline{M}(S)$. The reason for this is that although the ordinary points $pt\in M$ of a supermanifold tell us very little (namely, $|M|$) the $S$-points of $M$ tell us everything by the usual Yoneda argument. For example, in Appendix B.1 we explain how functions on a supermanifold are determined by their value at $S$-points, which immediately leads us to the correct notion of functions on generalized supermanifolds.

The supergeometry computations here owe some debt to investigations in [KS04], wherein *differential gorms* were defined and studied. The goals of the authors were somewhat different there, and computations were done in Grassmann coordinates. Below we do computations using the above functor of points approach. Of course, these two descriptions are entirely interchangeable when working with finite-dimensional supermanifolds. However, the second generalizes more readily; for example we can define field theories over mapping spaces of supermanifolds and stacks immediately. This feature is essential for understanding higher-dimensional field theories and gauged sigma models, respectively.

We will frequently use the parity reversal functor $\pi$. It has a few incarnations:

(1) for $A$ a (commutative) superalgebra, $\pi:\mathsf{Mod}_A\to\mathsf{Mod}_A$ takes a (left or right) $A$-module to the parity reversed (left or right) module;
(2) $\pi:\mathsf{SVBund}\to\mathsf{SVBund}$ takes a super vector bundle over a supermanifold to the parity reversed bundle; and
(3) $\pi:\mathsf{SVBund}\to\mathsf{SM}$ takes a super vector bundle to the total space of the parity reversed bundle.

When these distinctions matters we will be explicit.

1.9. **Outline of the Paper.** We have attempted to keep each of the sections below as self-contained as possible. The next section is the technical heart of the paper, where we carefully define $0|\delta$-$\mathsf{EFT}$s and begin to describe the homotopy functor gotten by taking concordance classes. In Section 3, we focus our attention on the $0|2$ sigma model, which allows us to define a quantization for $0|2$-$\mathsf{EFT}$s, proving Theorem 1.18. In Section 4, we combine the results of the previous two sections to prove the Chern-Gauss-Bonnet theorem. From Section 2.2.2 onward, $X$ is assumed to be an ordinary closed manifold.

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## 2. $0|\delta$-EFTs, Concordance and Integration

The goal of this section is to present and explain a careful definition of $0|\delta\text{-}\mathsf{EFT}$s; integrability considerations for the quantization will require a slight tweak on the presentation in [HKST09]. Next we provide convenient descriptions of $0|\delta\text{-}\mathsf{EFT}^\bullet(X)$ which will require some supergeometry computations: first to understand the functions on $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)$, and then to find the ones invariant under the Euclidean group. Then we give an algebraic characterization of when $0|\delta\text{-}\mathsf{EFT}^\bullet$s are concordant in Section 2.4. At the end of the section, we discuss integration on $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$.

2.1. **The Definition of $0|\delta\text{-}\mathsf{EFT}$s.** As described in the introduction, there are two pieces of data we need to construct a sequence $0|\delta$-Euclidean field theories in the sense of [HKST09]: a Euclidean group and a twist. So let

$$\underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta}) := \mathbb{R}^{0|\delta} \rtimes O(\delta),$$

where $\mathbb{R}^{0|\delta}$ acts by translations, and $O(\delta)$ by rotations. The homomorphism defining the twist,

$$\rho\colon \underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta}) \to \mathbb{Z}/2 \subset \mathbb{R}^\times,$$

is given by projection to the $O(\delta)$ factor, then application of the usual determinant homomorphism. Since for orthogonal matrices this lands in $\mathbb{Z}/2$, $0|\delta\text{-}\mathsf{EFT}^\bullet$ will define a 2-periodic sequence of functors.

*Remark* 2.1. The usual definition of Euclidean structure following [Fre99, ST11] would give a group $\mathbb{R}^{0|\delta} \rtimes \mathbb{Z}/2$, so strictly speaking the above is some kind of *extended* Euclidean structure. This alteration is motivated by considerations of so-called $R$-symmetry: the $0|2$-sigma model turns out to be invariant under a larger group than one would expect. $R$-symmetry can be interpreted as a remnant of dimensionally reduced theories. Indeed, our choice was originally motivated by the structures present in $1|2$-supersymmetric quantum mechanics implicit in [Wit82].

There is one technical modification we require in the definition. To motivate the discussion, we leap ahead a little: $0|2$ quantization wishes to integrate functions on $\mathcal{F}X := \underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)$. The problem is that even when $X$ is compact, $\mathcal{F}X$ might not be; for example,

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, S^1) \cong S^1 \times \mathbb{R}^{1|2}.$$

Thus, we need some control over the functions on $\mathcal{F}X$ in order for the integrals to converge; we will use a Gaussian measure, and so (roughly) we need to require functions to be polynomial in noncompact directions. One solution would be to single these out

$$\mathrm{Poly}\left(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)/\!/\, \underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta})\right) \subset C^\infty\left(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)/\!/\, \underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta})\right),$$

resulting in field theories being defined as functors with certain restrictions.[3] However, we prefer to modify $0|\delta\text{-}\mathsf{EB}(X)$ in such a way that *all* functors are field theories. A slight enlargement of the category of supermanifolds accomplishes this; namely we need to introduce a $\mathbb{Z}$-grading in addition to the usual super-grading.

[3]Indeed, the inclusion of the polynomials into all functions induces an isomorphism on concordance classes, so we can define the suspension map up to concordance by continuing down this road.

**Definition 2.2.** Let $V^{k|m}$ be a $\mathbb{N} \times \mathbb{Z}/2$-graded vector space, where forgetting the $\mathbb{N}$-grading produces a supervector space isomorphic to $\mathbb{R}^{k|m}$. An $\mathbb{N}$-*graded supermanifold* $X$ is a locally ringed (Hausdorff, second countable) space $(|X|, \mathcal{O}_X)$ whose structure sheaf is locally isomorphic to

$$\mathcal{O}_X \overset{\text{local}}{\cong} C^\infty(\mathbb{R}^n) \otimes \mathrm{Sym}^\bullet((V^{k|m})^*) \subset C^\infty(\mathbb{R}^{n+k}) \otimes \Lambda^\bullet((\mathbb{R}^m)^*)$$

as $\mathbb{N} \times \mathbb{Z}/2$-graded algebras. We say $X$ has super-dimension $(n,k)|m$, and will frequently call $X$ an $(n,k)|m$-supermanifold for short. As usual, morphisms between graded supermanifolds are morphisms of locally ringed spaces. We denote the resulting category by $\mathsf{grSM}$.

**Example 2.3.** Let $E^{k|m} \to X^n$ be a $k|m$-dimensional supervector bundle on a compact $n$-manifold $X$. The sheaf $\mathcal{O}_{\pi E} := \Gamma(X, \mathrm{Sym}^\bullet(E))$ gives an example of a $(n,k)|m$-supermanifold, where the $\mathbb{N}$-grading is from the polynomial degree in $\mathrm{Sym}^\bullet(E)$ (so the original fiber of the vector bundle is in degree $+1$). When $k = 0$, we get the usual definition of a supermanifold, but for $k \neq 0$ this captures the idea of a noncompact space with functions that are required to be polynomial in those noncompact directions.

**Example 2.4.** Consider $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$. There is a natural $\mathbb{R}$-action on the function on this space, gotten by dilation of $\mathbb{R}^{0|\delta}$. As will be shown below, the direct sum of the eigenspaces of this action returns a graded supermanifold, which is isomorphic to $\underline{\mathsf{grSM}}(\mathbb{R}^{0|\delta}, X)$, where the $\mathbb{N}$-grading on $C^\infty(\mathbb{R}^{0|\delta}) \cong \mathbb{R}[\theta_1, \dots, \theta_\delta]$ is by polynomial degree with $|\theta_i|$ having $\mathbb{N}$-grading $+1$, and (since $X$ is an ordinary manifold) $C^\infty X$ has degree zero, as required.

**Notation 2.5.** We can compute the inner hom $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$ in either the ordinary category of supermanifolds or the enlarged category $\mathsf{grSM}$, and we use the same notation in both cases due to claims made in the above example. However, by construction the functions on these will be quite different. As such we write $\mathcal{O}$ to denote the functions or sections of a line bundle computed in the category $\mathsf{grSM}$, and $C^\infty$ (respectively, $\Gamma$) to denote the algebra of functions (respectively, the space of sections) in the usual category of supermanifolds, $\mathsf{SM}$. We apply this same convention to $0|\delta\text{-}\mathsf{EB}(X)$. For $(n,0)|m$-supermanifolds the sheaves $\mathcal{O}$ and $C^\infty$ are canonically isomorphic, which will be utilized extensively in computations.

Using the argument reviewed in Appendix B.1, an element of $\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$ is a map of sets

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)(S) \to \mathcal{O}(S)$$

natural in $S$. We denote the set of such maps

$$\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))(S) \subset \mathsf{Hom}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)(S), \mathcal{O}(S))$$

and think of them as the set of values of functions evaluated at the point $S$. Similarly, an element of $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$ is a natural map

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)(S) \to C^\infty(S).$$

*Remark* 2.6. This remark discusses the relation between the categories $\mathsf{SM}$ and $\mathsf{grSM}$. Experts may be interested, but these facts have no real bearing on the content of this paper. Given an $(n,k)|m$-supermanifold, we can consider its $S$-points for $S$ an ordinary supermanifold. This functor on supermanifolds is representable, and gives a $(n+k)|m$-supermanifold (see Appendix B). We think of this as the ordinary supermanifold underlying the original $(n,k)|m$-supermanifold, where one obtains the former by a sort of $C^\infty$-completition of the latter. There is an inclusion of algebras of the polynomial functions on the $(n,k)|m$-supermanifold into the functions on the $(n+k)|m$-supermanifold, which induces a map of locally ringed spaces. The extra data contained by the $(n,k)|m$-supermanifold can be thought of as an affine structure on the noncompact directions. On the $(n+k)|m$-supermanifold this gives an action by the monoid $(\mathbb{R}, \times)$ whose nonzero eigenspaces are homogenous polynomials in the noncompact direction, i.e. the functions on the original $(n,k)|m$-supermanifold.

Saying this a little more carefully results in an equivalence of categories between supermanifolds with an action by $(\mathbb{R},\times)$ and the category of $(n,k)|m$-supermanifolds, as sketched in Appendix B. We thank Dmitry Roytenberg for explaining this to us.

We now give our definition of $0|\delta$ field theories.

**Definition 2.7.** A $0|\delta$-Euclidean field theory is a section

$$0|\delta\text{-}\mathsf{EFT}^\bullet(X) := \mathcal{O}(0|\delta\text{-}\mathsf{EB}(X), \mathcal{L}_\rho^\bullet)$$

of the line bundle defined above.

In particular,

$$0|\delta\text{-}\mathsf{EFT}^0(X) \cong \mathsf{Fun}^{\otimes}_{\mathsf{grSM}}(0|\delta\text{-}\mathsf{EB}(X), \underline{\mathbb{R}}).$$

*Remark* 2.8. The inclusion of polynomial functions into smooth ones induces maps

$$\mathsf{Fun}^{\otimes}_{\mathsf{grSM}}(0|\delta\text{-}\mathsf{EB}(X), \underline{\mathbb{R}}) \to \mathsf{Fun}^{\otimes}_{\mathsf{SM}}(0|\delta\text{-}\mathsf{EB}(X), \underline{\mathbb{R}}),$$

and we claim this is an isomorphism on concordance classes.

2.2. $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$ **and its Functions.** Now we wish to bring these abstract ideas down to earth in a few computations, starting with the familiar example of differential forms.

2.2.1. *Functions on* $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$*.* This classic case can be found in many places, for example [DEF+99, Kon03, HKST09]. We compute the $S$-points,

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)(S) \cong \{\Phi : S\times\mathbb{R}^{0|1}\to X\} \cong \{\Phi^* : C^\infty X \to C^\infty(S)\otimes C^\infty(\mathbb{R}^{0|1})\}.$$

Choosing a coordinate $\theta$ on $\mathbb{R}^{0|1}$ allows for a decomposition,

$$C^\infty(S)\otimes C^\infty(\mathbb{R}^{0|1}) \cong C^\infty S \oplus C^\infty S\cdot\theta$$

so we may express $\Phi$ in terms of the Taylor components

$$\Phi^* = f + \phi\theta.$$

Enforcing the condition that $\Phi^*$ be an algebra homomorphism we find $f : C^\infty X \to C^\infty S$ is a grade-preserving algebra homomorphism and $\phi : C^\infty X \to C^\infty S$ is a grade-reversing map that is an odd derivation with respect to $f$,

$$\phi(ab) = \phi(a)f(b) + (-1)^{p(a)} f(a)\phi(b), \quad a, b \in C^\infty(X).$$

But this is the standard description [DEF+99] of $\pi TX$ in terms of its $S$-points, which recovers the lemma

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X) \cong \pi TX.$$

Returning to the case that $\delta = 1$, we can define such functions in terms of $x \in C^\infty(X)$, by assigning their values on $S$-points as

$$x(\Phi) = f(x), \quad dx(\Phi) = \phi(x).$$

These are the zero- and one-forms in $\Omega^\bullet(X) \subset C^\infty(\pi TX)$, respectively. The polynomials generated by these give all forms on $X$ in the sense of [DEF+99]. If $X$ is an ordinary manifold, this gives all functions on $X$; however, if $X$ is an honest supermanifold, we need some $C^\infty$-completion of $\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X))$ to obtain all functions.

We wish to elaborate on this point a bit. When $X$ is a compact ordinary manifold, $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$ is compact as a supermanifold (which means its reduced manifold is compact), so considerations about $(n,k)|m$-supermanifolds do not apply: $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)) \cong \mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X))$. Said algebraically, since $\phi$ is an odd map, $dx(\Phi) = \phi(x) \in (C^\infty S)^{\mathrm{odd}}$, which implies that the function $dx$ is nilpotent (in fact square zero). Thus, smooth functions in $dx$ for different $x$ are always polynomial, and $C^\infty(\pi TX) \cong \mathcal{O}(\pi TX)$.

However, when $X^{n|m}$ is a supermanifold with $m \neq 0$, then $dx(\Phi)$ need not be nilpotent, so functions of $dx$ need not be polynomial; e.g., we can find $x$ so that the Taylor series of $e^{dx}$ does not truncate at some finite order. Said geometrically, $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$ contains a noncompact direction. Therefore, if we wish to integrate functions on $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$, we must do so with care. But this is exactly the case of 0|2-field theories, since

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X) \cong \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)) \cong \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \pi TX).$$

We now begin to tackle this problem.

2.2.2. *Functions on* $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$. For simplicity we will assume that $X$ is an ordinary closed manifold for the rest of the paper; the analysis for $X$ a genuine supermanifold is similar, but the extra signs clutter the formulas. Furthermore, homotopy invariants of a supermanifold $X$ are the same as those of $|X|$, so if we wish to do topology with $0|\delta$ field theories, we might as well assume that $X \cong |X|$.

An $S$-point of $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$ is a map $\Phi : S \times \mathbb{R}^{0|\delta} \to X$; we can take $S \in \mathsf{SM}$ or $S \in \mathsf{grSM}$ and a priori will get very different answers. However, since $C^\infty X \cong \mathcal{O}(X)$, the image of an algebra map $C^\infty X \to \mathcal{O}(S)$ will be contained in a $C^\infty$ subalgebra $\mathcal{O}(S)$, i.e., the degree zero polynomials of $\mathcal{O}(S)$. Therefore, to understand $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$ as an object of $\mathsf{grSM}$, it suffices to compute $S$-points in the category $\mathsf{SM}$. However, the functions on this inner hom will be different depending on if we compute them in $\mathsf{SM}$ or $\mathsf{grSM}$; we will discuss both of these computations after describing the sets $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)(S)$.

We choose coordinates $\{\theta_1, \dots, \theta_\delta\}$ on $\mathbb{R}^{0|\delta}$, which we think of as an isomorphism

$$C^\infty(S) \otimes C^\infty(\mathbb{R}^{0|\delta}) \cong C^\infty(S)[\theta_1, \dots, \theta_\delta],$$

with $\theta_i$ odd. The map $\Phi$ of supermanifolds is determined by the map $\Phi^* : C^\infty X \to C^\infty(S \times \mathbb{R}^{0|\delta})$ of superalgebras. We can express $\Phi^*$ in terms of its Taylor components,

$$\Phi^* = f + \sum_I \phi_I \theta_I,$$

where $I$ is a nonempty subset of $\{1, \dots, \delta\}$ and $f, \phi_I : C^\infty(X) \to C^\infty(S)$ are linear maps with restrictions that make $\Phi^*$ an algebra homomorphism. Notice that $f$ induces a map of supermanifolds $S \times \mathrm{pt} \to X$.

Given some $x \in C^\infty X$, we define a function $x \in \mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$ whose value at an $S$-point $\Phi$ is $x(\Phi) = f(x)$. This gives an inclusion of algebras

$$C^\infty X \hookrightarrow \mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)) \tag{5}$$

Other examples of functions are denoted by $d_I x$ for $x \in C^\infty X$, whose value at an $S$-point is defined as

$$(d_I x)(\Phi) := \phi_I(x).$$

We can form functions that are polynomial in the $d_I x$. However, we can not form arbitrary smooth functions $h$ of the $d_I x$ since these may not give rise to functions $h(d_I x)$ on $S$; for example when $S$ is the $(0,k)|m$ supermanifold determined by a point with the sheaf $\mathrm{Sym}(\mathbb{R}^{k|m})$, $k \neq 0$. We have proved the following.

**Lemma 2.9.** *The algebra* $\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$ *is generated by polynomials in* $d_I x$ *for* $x \in C^\infty X$ *and* $I$ *varying over all multi-indices* $\{i_1, \dots, i_k\}$.

*Remark* 2.10. The polynomials are called *differential gorms* in [KS04] for $\delta = 2$ and *differential worms* for higher $\delta$. From the above computations, $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$ is generated as an algebra by smooth functions in $x, d_I x$ for $|I|$ even, and polynomials in the $d_I x$ for $|I|$ odd.

We now consider the first nontrivial example of the above discussion, namely when $\delta = 2$.

**Example 2.11.** Consider the $S$-points with a choice of coordinate,

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)(S) \cong \{\Phi^* : C^\infty(X) \to C^\infty(S)[\theta_1, \theta_2]\},$$

which allows us to write Taylor components

$$\Phi^* = f + \phi_1\theta_1 + \phi_2\theta_2 + E\theta_1\theta_2,$$

where $\phi_i : C^\infty X \to (C^\infty S)^{\rm odd}$ and $f, E : C^\infty X \to (C^\infty S)^{\rm even}$. A computation shows that $\Phi^*$ being an algebra homomorphism requires

$$\begin{aligned} f(ab) &= f(a)f(b) \\ \phi_i(ab) &= \phi_i(a)f(b) - f(a)\phi_i(b) \quad i = 1, 2 \\ E(ab) &= E(a)f(b) + f(a)E(b) + \phi_1(a)\phi_2(b) + \phi_1(b)\phi_2(a). \end{aligned} \tag{6}$$

so $f$ is an algebra homomorphism, $\phi_1$ and $\phi_2$ are odd derivations with respect to $f$, and $E$ satisfies some more complicated identity. In Section 3.1 we give a (noncanonical) geometric characterization of this supermanifold, but for now we will work with the above algebraic one.

The functions on $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)$ are generated as an algebra by

$$x(\Phi) = f(x), \quad (d_1x)(\Phi) = \phi_1(x), \quad (d_2x)(\Phi) = \phi_2(x), \quad (d_2d_1x)(\Phi) = E(x).$$

We emphasize that in the world of $(n, k)|m$-supermanifolds, we can only take polynomials in the above functions, rather than arbitrary smooth functions. Unlike $d_1x$ and $d_2x$, notice that $d_2d_1x$ is not nilpotent, so this is a real restriction on $\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)) \subsetneq C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X))$.

2.3. **Group Actions on $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$.** Using the functor of points formalism, we compute actions of certain groups on $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$, most crucially the action of the Euclidean group, $\mathbb{R}^{0|\delta} \rtimes O(\delta)$, as this allows us to understand $0|\delta$ field theories over $X$. However, there is also an action by $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, \underline{\mathrm{Diff}}(X))$ that will be useful. Rather than giving an exhaustive characterization of these actions, we just explain the basics and give the minimal computations needed for the proofs below.

As in the previous subsection, we begin with the example of $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$.

2.3.1. *Actions on* $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$. The below computations give a supergeometric interpretation for familiar algebraic structures on differential forms. As mentioned above, the functions on $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$ are generated as an algebra by $x, dx$ for $x \in C^\infty(X)$. The action of the Euclidean group can be described by a $\mathbb{R}^{0|1}$-action and a $O(1) \cong \mathbb{Z}/2$-action. The infinitesimal generator of $\mathbb{R}^{0|1}$ acts on functions by odd derivations. Furthermore, a map $X \to Y$ induces a $\underline{\mathsf{Euc}}(\mathbb{R}^{0|1})$-equivariant map $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X) \to \underline{\mathsf{SM}}(\mathbb{R}^{0|1}, Y)$, so the action of this odd derivation must be natural with respect to pullback on the algebra of functions. As one would expect, the action on functions on $X$ is $x \mapsto dx$ and this extends uniquely to all differential forms by the Leibniz rule. The reflection, $\mathbb{Z}/2$, acts on functions by $dx \mapsto -dx$ and leaves $x$ fixed. We claim the infinitesimal action by

$$\mathrm{Lie}(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \underline{\mathrm{Diff}}(X))) \cong \pi T\, \underline{\mathrm{Vect}}(X) \cong \underline{\mathrm{Vect}}(X) \oplus \pi\, \underline{\mathrm{Vect}}(X)$$

is $v \in \underline{\mathrm{Vect}}(X)$ acts by the Lie derivative, $\mathcal{L}_v$, and $\psi \in \pi\, \underline{\mathrm{Vect}}(X)$ acts by interior multiplication, $\iota_\psi$. We see this by computing the composition that defines the action,

$$C^\infty X \stackrel{p^*}{\to} C^\infty S[\theta] \otimes C^\infty X \stackrel{\mathcal{G}^*}{\to} C^\infty S[\theta] \otimes C^\infty X \stackrel{\Phi^*}{\to} C^\infty(S)[\theta]$$

where $\mathcal{G}^* = v + \psi\theta$, $(v, \psi) \in \underline{\mathrm{Vect}}(X) \oplus \pi\, \underline{\mathrm{Vect}}(X)$, and $\Phi = f + \phi\theta$. Then we find on functions

$$(\mathcal{G}^*x)(\Phi) = f(vx) = (vx)(\Phi) \quad (\mathcal{G}^*dx)(\Phi) = \phi(vx) + f(\theta\psi x) = (\mathcal{L}_v dx)(\Phi) + (\iota_\psi dx)(\Phi).$$

Since $\mathcal{G}^* \in \mathrm{Lie}(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, \underline{\mathrm{Diff}}(X)))$ acts by derivations, the above formulas determine the action uniquely on $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)) \cong \Omega^\bullet(X)$. However, we're abusing notation here slightly, identifying

$$\theta\psi \in \mathbb{R}^{0|1} \otimes \pi(\underline{\mathrm{Vect}}(X)) \cong \underline{\mathrm{Vect}}(X)$$

and when we write $\iota_\psi$ (resp. $\theta\psi x$), we mean interior multiplication (resp. action by derivation) with the image of $\theta\psi$ under the above isomorphism.

The semidirect product structure between $\underline{\mathrm{Vect}}(X) \oplus \pi\, \underline{\mathrm{Vect}}(X)$ and $\underline{\mathfrak{euc}}(\mathbb{R}^{0|1})$ produces the usual Cartan identity,

$$[d, \iota_V] = \mathcal{L}_V.$$

The $\mathbb{Z}/2$-action shows that $d$ and $\iota_V$ are odd, and $\mathcal{L}_V$ is even.

2.3.2. *Actions on* $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$. Let $\mathcal{A} \in \underline{\mathrm{Diff}}(\mathbb{R}^{0|\delta})(S)$ and $\Phi \in \underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)(S)$. This means

$$\mathcal{A} : S \times \mathbb{R}^{0|\delta} \xrightarrow{\cong} S \times \mathbb{R}^{0|\delta}, \quad \Phi : S \times \mathbb{R}^{0|\delta} \to X,$$

and $\mathcal{A}$ is a map of bundles over $S$. By restricting $\mathcal{A} \in \underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta})(S) \subset \underline{\mathrm{Diff}}(\mathbb{R}^{0|\delta})(S)$, we get the action

$$\begin{array}{ccc} \underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)(S) \times \underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta}) & \to & \underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)(S) \\ \Phi, \mathcal{A} & \mapsto & \Phi \cdot \mathcal{A} = \Phi \circ \mathcal{A}. \end{array}$$

To understand

$$\begin{array}{ccc} \underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)(S) \times \underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, \underline{\mathrm{Diff}}(X))(S) & \to & \underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)(S) \\ \Phi, \mathcal{G} & \mapsto & \Phi \cdot \mathcal{G} \end{array}$$

geometrically, we think of $\mathcal{G}$ as an automorphism of the trivial bundle over $S \times \mathbb{R}^{0|\delta}$ with fiber $X$,

$$S \times \mathbb{R}^{0|\delta} \times X \xrightarrow{\mathcal{G}} S \times \mathbb{R}^{0|\delta} \times X,$$

and can turn $\Phi$ into a section of this bundle via

$$\mathrm{id} \times \Phi \in \Gamma(S \times \mathbb{R}^{0|\delta}, S \times \mathbb{R}^{0|\delta} \times X).$$

Then we define $\Phi \cdot \mathcal{G}$ as the composition

$$S \times \mathbb{R}^{0|\delta} \xrightarrow{\mathrm{id} \times \Phi} S \times \mathbb{R}^{0|\delta} \times X \xrightarrow{\mathcal{G}} S \times \mathbb{R}^{0|\delta} \times X \xrightarrow{p} X$$

where $p$ is projection.

Let $\underline{\mathfrak{euc}}(\mathbb{R}^{0|\delta})$ denote the Lie algebra of $\underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta})$. The infinitesimal action of odd an translation,

$$\vec{\alpha} \in \mathsf{Der}_{C^\infty S}(C^\infty(S) \otimes C^\infty(\mathbb{R}^{0|\delta})) < \underline{\mathfrak{euc}}(\mathbb{R}^{0|\delta})(S),$$

leads to an odd vector field on $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$. These act by derivations on $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$, and necessarily preserve $\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$ since we can consider this action as taking place in either the category $\mathsf{SM}$ or $\mathsf{grSM}$. The infinitesimal translations form an abelian Lie superalgebra (in particular these derivations square to zero) so we obtain a $\delta$-dimensional vector space, $\mathbb{R}^{0|\delta}$, that acts on $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$ by commuting differentials. Hence, the universal enveloping algebra,

$$\mathcal{U}(\mathbb{R}^{0|\delta}) \cong \mathrm{Sym}(\mathbb{R}^{0|\delta}),$$

acts on $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$ by differential operators where Sym denotes the graded symmetric product on super vector spaces. In fact, we claim the following.

**Proposition 2.12.** *Let $\mathcal{D}(\mathbb{R}^{0|\delta})$ denote the image of $\mathcal{U}(\mathbb{R}^{0|\delta})$ in differential operators on $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$. Then*

$$\{\mathcal{D}x \mid \mathcal{D} \in \mathcal{D}(\mathbb{R}^{0|\delta}),\ x \in C^\infty(X)\} \subset \mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))\}$$

*generates $\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}), X))$ as an algebra, where we have used the inclusion $C^\infty X \hookrightarrow \mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X))$ from Equation 5.*

*Proof.* As usual, we prove this with the functor of points; in showing there is a bijection of functions, we need to show there is a bijection of sets at every $S$-point,

$$C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))(S) := \mathsf{Hom}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)(S), C^\infty(S)) \cong \{\mathcal{D}x(\Phi) \mid \Phi \in \underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)(S)\}.$$

The action of the derivation $\vec{\alpha} \in \mathsf{Der}(C^\infty(S) \otimes C^\infty(\mathbb{R}^{0|\delta}))$ is by postcomposition,

$$d_\alpha \colon C^\infty(X) \xrightarrow{\Phi^*} C^\infty(S) \otimes C^\infty(\mathbb{R}^{0|\delta}) \xrightarrow{\vec{\alpha}} C^\infty(S) \otimes C^\infty(\mathbb{R}^{0|\delta}).$$

In lemma 2.9 we characterized $\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$ in terms of a choice of coordinates on $\mathbb{R}^{0|\delta}$. We compute the action of $d_\alpha$ given such a choice: let $\{\theta_1, \ldots, \theta_\delta\}$ be functions on $\mathbb{R}^{0|\delta}$, inducing an isomorphism

$$C^\infty(S) \otimes C^\infty(\mathbb{R}^{0|\delta}) \cong C^\infty(S)[\theta_1, \ldots, \theta_\delta].$$

We denote $d_i$ for the operator arising from unit infinitesimal translations in the $\theta_i$-direction, $1 \otimes \partial_{\theta_i} \in \mathsf{Der}(C^\infty(S) \otimes C^\infty(\mathbb{R}^{0|\delta}))$, and have

$$d_\alpha = \sum \alpha^i d_i.$$

for $\alpha^i \in C^\infty(S)$. Recall that $d_I x(\Phi) = \phi_I(x)$. We now consider the action of $d_i$ on $d_I x \in C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$; this is just composition. So

$$\sum \phi_I \theta_I \xrightarrow{d_i} \sum \phi_I \partial_{\theta_i} \theta^I \xrightarrow{d_I x} \phi_{i \bigcup I}(x)$$

and we have that $d_i(d_I x) = d_{i \cup I} x$ (perhaps with a sign from $\partial_{\theta_k} \theta^I$, depending on the chosen basis for the Taylor expansion). Given $I = \{i_1, \dots, i_k\}$, we can iterate the above action on the function $x \in C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$, finding

$$\mathcal{D}_I(x)(\Phi) = (d_I x)(\Phi) = \phi_I(x),$$

as claimed. □

This characterization of $\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$ makes it possible to describe the action by the Euclidean group. The action of odd translations $\mathbb{R}^{0|\delta}$ is the expected one given by composition,

$$\alpha \cdot (d_I x) = (d_\alpha d_I) x.$$

As expected, the action of $\mathbb{R}^{0|\delta}$ preserves $\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$. To compute the action of $O(\delta)$, observe that its action on the Lie superalgebra $\mathbb{R}^{0|\delta}$ naturally extends to one on $\mathcal{U}(\mathbb{R}^{0|\delta})$, the universal enveloping algebra of $\mathbb{R}^{0|\delta}$. Since the Lie superalgebra is abelian, this is the usual action of $O(\delta)$ on $\mathrm{Sym}(\mathbb{R}^{0|\delta})$.[4] Then $A \in O(\delta)$ acts on a function $\mathcal{D}x$ by

$$A(\mathcal{D}x) = (A\mathcal{D})x.$$

At this point, we can make some remarks.

*Remark* 2.13. The inclusions $i : \mathbb{R}^{0|1} \hookrightarrow \mathbb{R}^{0|\delta}$ give $\delta$-independent inclusions of $i^*\Omega^\bullet(X) \hookrightarrow \mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$. The map $i$ is a homomorphism of groups, so we get an induced action of the $\delta$-different de Rham differentials; this is exactly the action of the $d_i$. The above characterization of functions shows that $\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X))$ is freely generated by the commuting differentials $d_i$ and the functions $C^\infty(X)$. This is discussed in [KS04].

*Remark* 2.14. Since functions on $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$ contain $\delta$-order derivatives of functions on $X$, in some sense $0|\delta$-$\mathsf{EFT}(X)$ sees $\delta$-order geometric information about $X$. This should be no surprise, since $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X) \cong (\pi T)^\delta X$, the iterated odd tangent bundle.

**Example 2.15.** Now we consider the above in detail for $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)$. Functions are generated by

$$x(\Phi) = f(x), \quad d_1 x(\Phi) = \phi_1(x), \quad d_2 x(\Phi) = \phi_2(x), \quad d_2 d_1 x(\Phi) = E(x).$$

The odd translations, $\mathbb{R}^{0|2}$, act in the predictable way that was described before, using that $d_1 d_2 = -d_2 d_1$ and both $d_i$ square to zero. The rotations, $O(2)$ act via the usual 2-dimensional representation on the span of $d_1 x, d_2 x$; act trivially on $x$; and act by the determinant on $d_2 d_1 x$.

Next we wish to understand the action by

$$\mathrm{Lie}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, \underline{\mathrm{Diff}}(X))) \cong \pi T \pi T\, \underline{\mathrm{Vect}}(X) \cong \underline{\mathrm{Vect}}(X) \oplus \pi\, \underline{\mathrm{Vect}}(X) \oplus \pi\, \underline{\mathrm{Vect}}(X) \oplus \underline{\mathrm{Vect}}(X).$$

The algebraic computation of the action that parallels the geometric description at the beginning of the subsection is the composition,

$$C^\infty X \xrightarrow{p^*} C^\infty S[\theta_1, \theta_2] \otimes C^\infty X \xrightarrow{\mathcal{G}^*} C^\infty S[\theta_1, \theta_2] \otimes C^\infty X \xrightarrow{\Phi^*} C^\infty(S)[\theta_1, \theta_2],$$

where

$$\mathcal{G}^* = v + \psi_1 \theta_1 + \psi_2 \theta_2 + w \theta_1 \theta_2, \quad \Phi^* = f + \phi_1 \theta_1 + \phi_2 \theta_2 + E \theta_1 \theta_2,$$

[4] Among friends this is the action of the orthogonal group on the exterior algebra of the vector space $\mathbb{R}^\delta$.

where $v, w \in \underline{\mathrm{Vect}}(X)$, $\psi_1, \psi_2 \in \pi\underline{\mathrm{Vect}}(X)$. The action of $v$ is again by the Lie derivative, $\mathcal{L}_v$,

$$(\mathcal{L}_v x)(\Phi) = f(vx),\ (\mathcal{L}_v d_i x)(\Phi) = \phi(vx),\ (\mathcal{L}_v d_2 d_1 x)(\Phi) = E(vx).$$

Most of the action of $\psi_1$ and $\psi_2$ can be gleaned by considering inclusions $\mathbb{R}^{0|1} \hookrightarrow \mathbb{R}^{0|2}$: when restricting to the subspaces generated by $\{x, d_1 x\}$ or $\{x, d_2 x\}$, we get copies of the Cartan algebra. We denote the action of $\psi_i$ by $\iota_{\psi_i}$ and compute

$$\iota_{\psi_1} x(\Phi) = 0,\ \iota_{\psi_1} d_1 x(\Phi) = (\mathcal{L}_{\psi_1} x)(\Phi) = f(\epsilon\psi_1 x),\ \iota_{\psi_1} d_2 x(\Phi) = 0,\ \iota_{\psi_1} d_2 d_1 x(\Phi) = d_2 x(\Phi).$$

Similar formulas hold for the action of $\psi_2$. From the above, we notice that

$$[d_i, \iota_{\psi_j}] = \delta_{ij}\mathcal{L}_{\psi_j},$$

where $i, j$ are 1 or 2 and $\delta_{ij}$ is the usual delta function. Lastly, the action by $\iota_w$ is

$$\iota_w x = 0,\ \iota_w d_1 x = 0,\ \iota_w d_2 x = 0,\ (\iota_w d_2 d_1 x)(\Phi) = f(wx) = (wx)(\Phi).$$

Finally, we note that identity

$$[d_1, [d_2, \iota_w]] = \mathcal{L}_w.$$

**Example 2.16.** We need to say just a little about certain operators in the general case, $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$. As mentioned previously we have $\delta$ independent differentials. Let $\Delta$ be the operator which is a composition of all of these, or equivalently the image in differential operators of the top tensor power of $\mathcal{U}(\mathbb{R}^{0|\delta})$ (up to a nonzero scalar). We note that $\Delta$ will not act by derivations. Since $\Delta$ lies in the top power of $\mathrm{Sym}(\mathbb{R}^{0|\delta})$, we see that $O(\delta)$ acts by the determinant on $\Delta$, so in particular functions of the form $\Delta x$ will be odd with respect to the field theory twist explained previously.

In understanding the action of $\mathcal{G} \in \underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, \underline{\mathrm{Vect}}(X))$, we will be concerned with elements

$$\mathcal{G}_v = v, \quad \mathcal{G}_w = \theta_1 \dots \theta_\delta w.$$

An identical computation to the above shows that $\mathcal{G}_v$ acts by the Lie derivative, $\mathcal{L}_v$, and we denote the action of $\mathcal{G}_w$ by $\iota_w$. A computation shows

$$[d_\delta, \dots, [d_2, [d_1, \iota_w]] \dots] = \mathcal{L}_w.$$

*Remark* 2.17. The action by the Euclidean group is natural in $X$ since a map $X \to Y$ induces a $\underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta})$-equivariant map $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X) \to \underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, Y)$. This generalizes the usual naturality of de Rham $d$. In particular, $\Delta$ acts naturally, which will be important in the next subsection. Another way to package this informtation is that a map $X \to Y$ induces a functor of groupoids, $0|\delta\text{-}\mathsf{EB}(X) \to 0|\delta\text{-}\mathsf{EB}(Y)$.

### 2.4. Concordance Classes of $0|\delta$-EFTs.

Throughout, we assume that $X$ is compact. The following proposition is the key to understanding concordance classes of $0|\delta$-$\mathsf{EFT}$s.

**Proposition 2.18.** *Two $0|\delta$-Euclidean field theories over $X$ are concordant if and only if they are $\Delta$-cohomologous:*

$$\mathsf{E}_0 \sim \mathsf{E}_1 \quad \Longleftrightarrow \quad \mathsf{E}_0 - \mathsf{E}_1 = \Delta e.$$

*Proof of Proposition.* One direction of this is easy. If $\psi_1 - \psi_0 = \Delta\alpha$, then define

$$\Psi = \psi_0 + \Delta(\lambda\alpha).$$

Since $\underline{\mathsf{Euc}}(\mathbb{R}^{0|2})$ acts on $\psi_0$ and $\Delta\alpha$ trivially or via det, it acts on $\Psi$ in the same way, so $\Psi$ is a field theory of the appropriate twist. Furthermore, $\Psi$ pulls back correctly under $i_0^*$ and $i_1^*$, so we get a concordance.

Now say that $\Psi$ is a concordance from $\psi_1$ to $\psi_0$, and let $\partial_\lambda$ be a nonvanishing vector field on $\mathbb{R}$. In the examples above, we described an operator $\iota_{\partial_\lambda}$ satisfying a Cartan-type formula,

$$[d_\delta, \dots, [d_2, [d_1, \iota_{\partial_\lambda}]] \dots] = \mathcal{L}_{\partial_\lambda},$$

where $\mathcal{L}_{\partial_\lambda}$ is the Lie derivative. With this in hand, we are able to run the usual Stokes-type argument; namely we define a degree 1 linear map

$$Q : 0|\delta\text{-}\mathsf{EFT}^\bullet(X \times \mathbb{R}) \to 0|\delta\text{-}\mathsf{EFT}^{\bullet-1}(X)$$

with the property

$$\Delta Q = i_1^* - i_0^*,$$

and so $Q(\Psi)$ will produce $\alpha$. So let

$$Q(\Psi) := \int_0^1 i_\lambda^* \iota_{\partial_\lambda} \Psi d\lambda,$$

where the integral is of a $0|2\text{-}\mathsf{EFT}^\bullet(X)$-valued function on $\mathbb{R}$. Using that $d_k\Psi = 0$ for all $k$, we find

$$\mathcal{L}_{\partial_\lambda}\Psi = [d_\delta, \dots, [d_2, [d_1, \iota_{\partial_\lambda}]] \dots]\Psi = \Delta\iota_{\partial_\lambda}\Psi.$$

We calculate

$$\Delta Q\Psi = \int_0^1 i_\lambda^* \Delta\iota_{\partial_\lambda}\Psi d\lambda = \int_0^1 i_\lambda^* \mathcal{L}_{\partial_\lambda}\Psi d\lambda = i_1^*\Psi - i_0^*\Psi,$$

where the first equality is differentiation under the integral together with naturality of $\Delta$, and the last is the fundamental theorem of calculus. Thus, we have shown that $\psi_1$ and $\psi_0$ are $\Delta$-cohomologous. $\square$

Now we can prove the key Lemma for showing partition functions are manifold invariants.

*Proof of Corollary 1.16.* We compute

$$0|2\text{-}\mathsf{EFT}^{\mathrm{ev}}(\mathrm{pt}) \cong C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, \mathrm{pt})/\!/\, \underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta})) \cong C^\infty(\mathrm{pt}\,/\!/\, \underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta})) \cong C^\infty(\mathrm{pt})^{\underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta})} \cong \mathbb{R},$$

and by a similar computation

$$0|\delta\text{-}\mathsf{EFT}^{\mathrm{odd}}(\mathrm{pt}) = \{0\}.$$

Now, the action of $\underline{\mathsf{Euc}}(\mathbb{R}^{0|\delta})$ on pt is trivial, so the action on functions is also trivial. Hence if $E_0$ and $E_1$ are concordant, we have

$$E_0 - E_1 = \Delta e = 0 \implies E_0 = E_1,$$

since $\Delta$ applied to a (constant) function $e$ is necessarily zero. $\square$

2.5. **Integration on $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$.** Integration of functions on $\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)$ is particularly easy, owing to the canonically trivialized Berezinian line. Explicitly, integration is the composition

$$C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|1}, X)) \cong \Omega^\bullet(X) \xrightarrow{\text{project}} \Omega^{\mathrm{top}}(X) \xrightarrow{\int} \mathbb{R},$$

where the last arrow requires an orientation on $X$. We claim that a similar situation holds for $\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$. The key result is the following.

**Proposition 2.19.** *Given a choice of connection on $X$, there is an isomorphism*

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X) \cong \pi T(T^{\delta-1}X)$$

*as supermanifolds. For $\delta > 2$ this isomorphism requires a framing of $\mathbb{R}^{0|\delta}$.*

*Remark* 2.20. We emphasize that the $\mathbb{R}$-action on $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)$ is not preserved by the above isomorphism.

With this proposition, we can define integration (for compactly supported or Schwartz functions) as

$$C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X)) \cong \Omega^\bullet(T^{\delta-1}X) \xrightarrow{\text{project}} \Omega^{\mathrm{top}}(T^{\delta-1}X) \xrightarrow{\int} \mathbb{R}.$$

Furthermore, since $TM$ is canonically oriented for any manifold $M$, this integration map has no topological obstruction on $X$.

*Proof of Proposition 2.19.* First we prove the proposition for $\delta = 2$. A connection on $X$ splits $T(TX)$ into horizontal and vertical subspaces,

$$T(TX) \cong H(TX) \oplus V(TX) \cong p^*(TX \oplus TX)$$

where we get the second isomorphism from maps $Tp : H(TX) \to TX$ and the canonical map $V(TX) \to TX$. Sprinkling in the parity reversal functor we get

$$\pi T(TX) \cong p^*(\pi(TX \oplus TX)) \cong \underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)$$

where the second isomorphism uses Lemma 1.20 (which will be proved in Section 3.1) concluding the proof for $\delta = 2$.

Now we iterate the above isomorphism,

$$\underline{\mathsf{SM}}(\mathbb{R}^{0|\delta}, X) \cong (\pi T)^\delta X \cong \pi T(T^{\delta-1}X),$$

where the first isomorphism requires a framing on $\mathbb{R}^{0|\delta}$. □

*Remark* 2.21. Following [KS04], we can show that the section

$$d_1x^1 d_1\xi^1 \cdots d_1x^n d_1\xi^n d_2x^1 d_2\xi^1 \cdots d_2x^n d_2\xi^n \in \Gamma(\mathrm{Ber}(T\,\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, U))$$

is independent of the choice of coordinates, verifying that the Berezinian of $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, U)$ is canonically trivialized independent of the choice of connection used above. However, since the integration map from the sigma model uses the metric and connection to define a Gaussian measure, we prefer the more geometric argument above. One can check (e.g., in coordinates) that the two sections of the Berezinian are in fact equal.

## 3. Quantization and the 0|2-Sigma Model

Below we wish to define quantization for $0|2$-$\mathsf{EFT}^\bullet$s. As described in the introduction, we interpret this in terms of a Lagrangian field theory with space of fields

$$\mathcal{F}X := \underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X).$$

The action function, $\mathcal{S} \in C^\infty(\mathcal{F}X)$, will depend on a choice of metric and potential function on $X$. Subsequently we will show that $e^{-\mathcal{S}}\frac{\mathcal{D}\Phi}{N}$ determines a Gaussian measure on $\mathcal{F}$. The language of field theories would motivate us to compute expectation values of observables $\omega \in \mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X))$, which are

$$\langle\omega\rangle = \int_{\mathcal{F}X} \omega e^{-\mathcal{S}} \frac{\mathcal{D}\Phi}{N}.$$

If it happens that $\omega$ is invariant under the action of the Euclidean group on $\mathcal{F}$, $\langle\omega\rangle$ enjoys certain rigidity properties, via Lemma 1.13. This fact will be critical in the proof of the Chern-Gauss-Bonnet theorem.

In this section we first give a geometric characterization of the space of fields via Lemma 1.20; these are called the *component fields* in [Fre99]. Then we define the action functional and express it in terms of these components, proving Lemma 1.21. Finally, we verify that this action does indeed define a Gaussian measure satisfying Definition 1.12, thereby proving Theorem 1.18.

For a nice introduction to supersymmetric Lagrangian field theories, see [Fre99]; for further details and for an emphasis on the functor of points approach, see "Homework" in [DEF$^+$99].

### 3.1. **Component Fields.**

In this section we prove Lemma 1.20 by giving a bijection on $S$-points,

$$p^*\pi(TX \oplus TX)(S) \cong \underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)(S).$$

To define the map, there is some preliminary work to be done. Recall that the ordinary covariant Hessian is a map over $X$

$$\mathrm{Hess} : TX \otimes TX \to \mathsf{Diff}^{\leq 2}(X)$$

that takes pairs of tangent vectors and outputs a second order differential operator. We can also define the Hessian on pairs of odd tangent vectors, via the isomorphism

$$\pi TX \otimes \pi TX = (\mathbb{R}^{0|1} \otimes TX) \otimes (\mathbb{R}^{0|1} \otimes TX) \overset{\sigma}{\cong} (\mathbb{R}^{0|1} \otimes \mathbb{R}^{0|1}) \otimes (TX \otimes TX) \cong TX \otimes TX$$

where we recall that $\pi TX := \mathbb{R}^{0|1} \otimes TX$ and $\mathbb{R}^{0|1} \otimes \mathbb{R}^{0|1} \cong \mathbb{R}$; $\sigma$ denotes the braiding isomorphism. Precomposing Hess with the above gives an even map over $X$,

$$\text{Hess} : \pi TX \otimes \pi TX \to \mathsf{Diff}^{\leq 2}(X). \tag{7}$$

Given an $S$-point $f : S \to X$, we can pull back to obtain a map over $S$

$$f^* \text{Hess} : (f^*\pi TX) \otimes (f^*\pi TX) \to f^*\mathsf{Diff}^{\leq 2}(X).$$

Recall that an $S$-point of $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)$ is a quadruple $(f, \phi_1, \phi_2, E)$, where

$$\Phi^* = f + \phi_1\theta_1 + \phi_2\theta_2 + E\theta_1\theta_2, \quad \Phi^* \in \mathsf{ALG}(C^\infty X, C^\infty S[\theta_1, \theta_2]). \tag{8}$$

We can plug $\phi_1$ and $\phi_2$ into the above map and get

$$(f^* \text{Hess})(\phi_1, \phi_2) \in \Gamma(f^*\mathsf{Diff}^{\leq 2}(X)).$$

We note $S$-points of $\mathsf{Diff}^{\leq 2}(X)$ are maps of vector spaces $C^\infty X \to C^\infty S$ satisfying some additional conditions. Explicitly, on $X$ there is the evaluation map

$$\Gamma(\mathsf{Diff}^{\leq 2}(X)) \otimes_{\mathbb{R}} C^\infty X \to C^\infty X,$$

which is a map of sheaves of $C^\infty X$-modules via the left action of $C^\infty X$ on differential operators. Using the map $f^* \colon C^\infty X \to C^\infty S$, we obtain a map of sheaves of $C^\infty S$-modules

$$C^\infty(S) \otimes_{f^*} \Gamma(f^*\mathsf{Diff}^{\leq 2}(X)) \otimes_{\mathbb{R}} C^\infty(X) \to C^\infty(S) \otimes_{f^*} C^\infty(X) \cong C^\infty(S).$$

So in particular, given $f^* \text{Hess}(\phi_1, \phi_2) \in \Gamma(f^*\mathsf{Diff}^{\leq 2}(X))$ and $x \in C^\infty X$, we get a function in $C^\infty(S)$.

**Lemma 3.1.** *Let $(f, \phi_1, \phi_2, E)$ be an $S$-point of $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)$. Then*

$$(f, \phi_1, \phi_2, E - (f^* \text{Hess})(\phi_1, \phi_2))$$

*is an $S$-point of $p^*\pi(TX \oplus TX)$. Equivalently, $F := E - (f^* \text{Hess})(\phi_1, \phi_2))$ is an even derivation with respect to $f$.*

*Proof of Lemma 1.20 using Lemma 3.1.* The map in 3.1 is natural in $S$, since morphisms of sheaves are well-behaved under pull-backs. This gives the required natural map on the functor of points. In addition, the map is invertible, so implies Lemma 1.20.□

*Proof of Lemma 3.1.* The proof follows from direct computation. We assume that $X$ is an ordinary manifold, though a similar result holds with some extra signs for a general supermanifold target.

First we note that the Hessian is $C^\infty X$-linear in both vectors, and so is a map of sheaves of $C^\infty X$-modules. We have the formula[5]

$$\text{Hess}(\phi_1, \phi_2)(ab) = (\text{Hess}(\phi_1, \phi_2)a) \cdot b + a \cdot \text{Hess}(\phi_1, \phi_2)b + \phi_1(a) \cdot \phi_2(b) + \phi_1(b) \cdot \phi_2(a).$$

on $X$, and so when we pull back the Hessian to $S$, for $\phi_1, \phi_2 \in \Gamma(f^*\pi TX)$, and $a, b \in C^\infty X$ we find

$$\begin{aligned} f^* \text{Hess}(\phi_1, \phi_2)(ab) \quad &= \quad (f^* \text{Hess}(\phi_1, \phi_2)(a)) \cdot f(b) + f(a) \cdot f^* \text{Hess}(\phi_1, \phi_2)(b) \\ &\quad\ + \phi_1(a) \cdot \phi_2(b) + \phi_1(b) \cdot \phi_2(a), \end{aligned}$$

where both sides are elements of $C^\infty S$. The above argument is the functor of points version that Hess—being a tensor—is determined by its value (and well-defined) at points.

For convenience we recall the conditions for $E$ to be a component of an $S$-point of $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)$:

$$E(ab) = E(a)f(b) + f(a)E(b) + \phi_1(a)\phi_2(b) + \phi_1(b)\phi_2(a).$$

[5]This can be verified directly with the classical formula for the Hessian, $v \otimes w \mapsto vw - \nabla_v w$, with some care not to introduce extra signs from the braiding isomorphism, $\sigma$.

Upon subtracting, $F := E - f^* \operatorname{Hess}(\phi_1, \phi_2)$ is thus an even derivation,

$$\begin{aligned}(E + f^* \operatorname{Hess}(\phi_1,\phi_2))(ab) &= E(a)f(b) + (f^* \operatorname{Hess}(\phi_1,\phi_2)(a))f(b) \\ &\quad + f(a)E(b) + f(a)f^* \operatorname{Hess}(\phi_1,\phi_2)(b) \\ &= F(a)f(b) + f(a)F(b).\end{aligned}$$

This completes the proof. □

*Remark* 3.2. Lemma 1.20 also allows us to say something about spaces of fields for other $d|2$ field theories. Notice

$$\underline{\mathsf{SM}}(\Sigma^{d|2}, X) \cong \underline{\mathsf{SM}}(\Sigma^{d|0}, \underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)),$$

where we've assumed the odd plane bundle on $\Sigma$ is the topologically trivial one. Thus, for a sigma model with fields

$$\mathcal{F} := \underline{\mathsf{SM}}(\Sigma^{d|2}, X),$$

the component fields are quadrupoles

$$f : \Sigma^d \to X,\ \phi_1 \in \Gamma(f^*\pi TX),\ \phi_2 \in \Gamma(f^*\pi TX),\ F \in \Gamma(f^*TX). \tag{9}$$

When $d = 2$, this fact appears to be well-known—at least to physicists—but to the author's knowledge Lemma 1.20 is the first rigorous proof for a non-linear target. Notice also that we can restate the data of 9 as

$$f \in \underline{\mathsf{SM}}(\Sigma, X),\ \phi_1 \in \pi T_f(\underline{\mathsf{SM}}(\Sigma, X)),\ \phi_2 \in \pi T_f(\underline{\mathsf{SM}}(\Sigma, X)),\ F \in T_f(\underline{\mathsf{SM}}(\Sigma, X)).$$

This shows that Lemma 1.20 holds for the generalized manifolds that are mapping spaces. To extend to all generalized manifolds would require a notion of Hessian.

3.2. **The Lagrangian Density.** As is usual in Lagrangian mechanics the action functional is defined in terms of a Lagrangian density, which we will state in terms of $S$-points,

$$\mathcal{S}(\Phi) = \int_{S\times\mathbb{R}^{0|2}/S} \mathcal{L}, \quad \Phi \in \underline{\mathsf{SM}}(S \times \mathbb{R}^{0|2}, X),$$

where $\mathcal{L} \in \operatorname{Ber}(S\times\mathbb{R}^{0|2}/S)$ is a relative density; see Appendix A for details. The Lagrangian will contain two parts,

$$\mathcal{L} := \frac{1}{2}\|T\Phi\|^2 - \lambda\Phi^* h \cdot \operatorname{Ber}$$

for $h \in C^\infty X$ and $\operatorname{Ber} \in \Gamma(\operatorname{Ber}(S \times \mathbb{R}^{0|2}/S))$. We think of $\|T\Phi\|^2$ as the kinetic energy of the map $\Phi$ and $\Phi^*h$ as its potential energy. In this section we focus on defining and understanding the kinetic part. We will give a coordinate-independent definition of $\mathcal{L}$ in the $0|2$-setting, and then with a choice of coordinates set things up for computations that follow.

Let $\Phi \in \underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)(S)$. Then $T\Phi \in \Gamma(S \times \mathbb{R}^{0|2}, \mathsf{Hom}_{S\times\mathbb{R}^{0|2}}(T\mathbb{R}^{0|2}, \Phi^*TX))$, where (in an abuse of notation) $T\mathbb{R}^{0|2}$ denotes the vertical tangent bundle to $S \times \mathbb{R}^{0|2} \to S$. The metric on $X$ gives a pairing

$$\langle - \rangle \colon \Phi^*TX \otimes \Phi^*TX \to C^\infty(S \times \mathbb{R}^{0|2}),$$

which we apply to $T\Phi \otimes T\Phi \in \Gamma(S \times \mathbb{R}^{0|2}, \mathsf{Hom}_{S\times\mathbb{R}^{0|2}}(T\mathbb{R}^{0|2}, \Phi^*TX)^{\otimes 2})$ to obtain

$$\|T\Phi\|^2 := \langle T\Phi \otimes T\Phi\rangle \in \Gamma(S \times \mathbb{R}^{0|2}, \mathsf{Hom}_{S\times\mathbb{R}^{0|2}}(T\mathbb{R}^{0|2} \otimes T\mathbb{R}^{0|2}, \mathbb{R}))$$

where $\mathbb{R}$ is the trivial bundle on $S\times\mathbb{R}^{0|2}$. By the symmetry of the pairing $\langle - \rangle$, we find that

$$\|T\Phi\|^2 \in \Gamma(\operatorname{Sym}^2((T\mathbb{R}^{0|2})^*) \subset \Gamma((T\mathbb{R}^{0|2} \otimes T\mathbb{R}^{0|2})^*) \cong \Gamma(\mathsf{Hom}_{S\times\mathbb{R}^{0|2}}(T\mathbb{R}^{0|2} \otimes T\mathbb{R}^{0|2}, \mathbb{R}))$$

where $\operatorname{Sym}^2((T\mathbb{R}^{0|2})^*)$ is the second symmetric power of the super vector bundle $(T\mathbb{R}^{0|2})^*$. This bundle is precisely $\operatorname{Ber}(S \times \mathbb{R}^{0|2}/S)$, verifying that $\|T\Phi\|^2$ is indeed a section of the relative Berezinian.

If we follow the action of $\underline{\mathsf{Euc}}(\mathbb{R}^{0|2})$ through the definition of $\|T\Phi\|^2$, we find that it acts on the map entirely through its action on $\operatorname{Sym}^2((T\mathbb{R}^{0|2})^*)$, which in turn is induced from from the action of $\underline{\mathsf{Euc}}(\mathbb{R}^{0|2})$ on $\mathbb{R}^{0|2}$. To be explicit, the action of translations is trivial, and given an $S$-point $A$ of $O(2)$, it acts by $1/\det(A)$ on the bundle $\mathrm{Sym}^2((T\mathbb{R}^{0|2})^*)$. This is the identical action to that of $\underline{\mathsf{Euc}}(\mathbb{R}^{0|2})$ on $\mathrm{Ber}(S\times\mathbb{R}^{0|2}/S)$, so the map

$$\|T\Phi\|^2 : \underline{\mathsf{SM}}(\mathbb{R}^{0|2},X)\to \mathrm{Ber}(S\times\mathbb{R}^{0|2}/S)$$

is $\underline{\mathsf{Euc}}(\mathbb{R}^{0|2})$-equivariant, and so is an *invariant* $\mathrm{Ber}(S\times\mathbb{R}^{0|2}/S)$-valued function.

Above we understood $\|T\Phi\|^2$ in terms of the supergeometry of $\underline{\mathsf{SM}}(\mathbb{R}^{0|2},X)$; presently we wish to describe $\|T\Phi\|^2$ via local geometry on $X$ (e.g., curvature and the Riemannian metric). This will involve a small detour wherein we give an algebraic description of $\|T\Phi\|^2$.

First we trivialize the Berezinian by chosing coordinates $\theta_1,\theta_2$, which gives us trivializing sections $\partial_{\theta_1},\partial_{\theta_2}$ of $T\mathbb{R}^{0|2}$, so

$$\Phi \quad \overset{\|T\Phi\|^2}{\mapsto} \quad \langle T\Phi(\partial_{\theta_1}),T\Phi(\partial_{\theta_2})\rangle[d\theta_1 d\theta_2].$$

Below we will focus on computing $\langle T\Phi(\partial_{\theta_1}),T\Phi(\partial_{\theta_2})\rangle$. We require the following elementary result.

**Lemma 3.3.** *Let $f:N\to M$ be a map of supermanifolds. Then*

$$\mathsf{Der}(C^\infty M, C^\infty M)\otimes_f C^\infty N\cong \mathsf{Der}_f(C^\infty M, C^\infty N).$$

*Proof.* For $W\otimes n\in\mathsf{Der}(C^\infty M,C^\infty M)\otimes_f C^\infty N$ we define a map

$$W\otimes n\mapsto n\cdot V,\quad V(m):=(f^*W)m.$$

One can show that map is bijective abstractly, but we will need an explicit inverse map for computations below. As usual with maps *into* a tensor product, this inverse is somewhat less natural and we need coordinates $\{x^i\}$ on $M$ to define it. We will show the above isomorphism holds in each coordinate patch, and the sheaf property (or partitions of unity) will prove the result.

With a choice of coordinates in effect, given $V\in\mathsf{Der}_f(C^\infty M,C^\infty N)$, we get a map

$$V\mapsto\sum(\partial_{x^i})\otimes_f V(x^i).$$

One can check explicitly that this defines an inverse in the given chart $\{x^i\}$. □

To calculate $\langle T\Phi(\partial_{\theta_1}),T\Phi(\partial_{\theta_2})\rangle$, we apply Lemma 3.3 to $N=S\times\mathbb{R}^{0|2}$ and $M=X$. For an $S$-point $\Phi\in\underline{\mathsf{SM}}(\mathbb{R}^{0|2},X)(S)$, we will examine the composition

$$\begin{array}{c}
\Big(\mathsf{Der}(C^\infty X,C^\infty(S\times\mathbb{R}^{0|2}))\Big)\bigotimes_{C^\infty(S\times\mathbb{R}^{0|2})}\Big(\mathsf{Der}(C^\infty X,C^\infty(S\times\mathbb{R}^{0|2}))\Big)\\
\downarrow\cong\\
\Big(\mathsf{Der}(C^\infty X)\otimes_\Phi C^\infty(S\times\mathbb{R}^{0|2})\Big)\bigotimes_{C^\infty(S\times\mathbb{R}^{0|2})}\Big(\mathsf{Der}(C^\infty X)\otimes_\Phi C^\infty(S\times\mathbb{R}^{0|2})\Big)\\
\downarrow\cong\\
\mathsf{Der}(C^\infty X)\otimes_{C^\infty X}\mathsf{Der}(C^\infty X)\otimes_\Phi C^\infty(S\times\mathbb{R}^{0|2})\\
\downarrow g\\
C^\infty X\otimes_\Phi C^\infty(S\times\mathbb{R}^{0|2})\\
\downarrow \mathrm{act}\\
C^\infty(S\times\mathbb{R}^{0|2})
\end{array}$$

where in the last line we use the action of $C^\infty X$ on $C^\infty(S\times\mathbb{R}^{0|2})$ by $\Phi^*$, and in the second to last line the metric on $X$ is thought of as

$$g\colon\mathsf{Der}(C^\infty X)\otimes_{C^\infty X}\mathsf{Der}(C^\infty X)\to C^\infty X.$$

Now we compute for an $S$-point $\Phi:S\times\mathbb{R}^{0|2}\to X$,

$$T\Phi(\partial_{\theta_1})=\psi_1+\theta_2F,\quad T\Phi(\partial_{\theta_2})=\psi_2-\theta_1F,$$

so that $T\Phi(\partial_{\theta_i})\in\mathrm{Der}_\Phi(C^\infty X,C^\infty(S\times\mathbb{R}^{0|2}))$. Lemma 3.3 gives us an isomorphism

$$T\Phi(\partial_{\theta_i})\in\mathrm{Der}(C^\infty X,C^\infty(S\times\mathbb{R}^{0|2}))\cong\mathrm{Der}(C^\infty X,C^\infty X)\otimes_\Phi C^\infty(S\times\mathbb{R}^{0|2})$$

and using the proof of the lemma we find

$$T\Phi(\partial_{\theta_1}) \mapsto \sum_i \frac{\partial}{\partial x^i} \otimes_\Phi (d_1 x^i + \theta_2 d_2 d_1 x^i)(\Phi),$$
$$T\Phi(\partial_{\theta_2}) \mapsto \sum_j \frac{\partial}{\partial x^j} \otimes_\Phi (d_2 x^j - \theta_1 d_2 d_1 x^j)(\Phi),$$

where as usual we are identifying functions with their natural transformations. Then we can apply the pairing $g$,

$$\langle T\Phi(\partial_{\theta_1}), T\Phi(\partial_{\theta_2})\rangle = \sum_{ij} g_{ij} \otimes_\Phi (d_1 x^i + \theta_2 d_2 d_1 x^i)(d_2 x^j - \theta_1 d_2 d_1 x^j)(\Phi)$$

where $g_{ij}$ are the components of $g$ in the given coordinates.

So now we need to understand how the pulled back metric, $\Phi^* g_{ij}$, acts on functions on $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, M)$. So we compute $\Phi^*(g_{ij})$, getting

$$\Phi^*(g_{ij}) = g_{ij}(\Phi) + \theta_1 d_1 g_{ij}(\Phi) + \theta_2 d_2 g_{ij}(\Phi) + \theta_1\theta_2 d_2 d_1 g_{ij}(\Phi).$$

Putting this together we obtain an element of $C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, M))$, whose value at an $S$-point $\Phi$ is

$$\begin{aligned}\langle T\Phi(\partial_{\theta_1}), T\Phi(\partial_{\theta_2})\rangle \;=\; \sum_{i,j} \Big( &(g_{ij} + \theta_1 d_1 g_{ij} + \theta_2 d_2 g_{ij} + \theta_1\theta_2 d_2 d_1 g_{ij})(\Phi) \\ &\cdot (d_1 x^i + \theta_2 d_2 d_1 x^i)(d_2 x^j - \theta_1 d_2 d_1 x^j)(\Phi)\Big). \end{aligned} \tag{10}$$

This horrendous formula will simplify drastically and have more obvious geometric meaning when we pass from the Lagrangian density to the action functional.

3.3. **The Action Functional.** In this section we prove Lemma 1.21. We accomplish this by computing the integral

$$\mathcal{S}(\Phi) := \int_{S\times\mathbb{R}^{0|2}/S} \frac{1}{2}\|T\Phi\|^2 = \int_{S\times\mathbb{R}^{0|2}/S} \frac{1}{2}\langle T\Phi(\partial_{\theta_1}), T\Phi(\partial_{\theta_2})\rangle [d\theta_1 d\theta_2]$$

using equation 10. Recall that

$$\int_{S\times\mathbb{R}^{0|2}/S} \theta_1\theta_2 [d\theta_1 d\theta_2] = 1 \in C^\infty(S),$$

so if we expand equation 10 and project to the $\theta_1\theta_2$ component, we get

$$\mathcal{S}(\Phi) \;=\; \frac{1}{2}\sum_{i,j} (g_{ij} d_2 d_1 x^i d_2 d_1 x^j + d_1 g_{ij} d_2 d_1 x^i d_2 x^j + d_2 g_{ij} d_1 x^i d_2 d_1 x^j + d_2 d_1 g_{ij} d_1 x^i d_2 x^j)(\Phi).$$

Using Lemma 1.20, we can interpret the above in terms of more familiar Riemmanian geometry of $X$. First notice that for $x^i$ a coordinate, $\mathrm{Hess}(\phi_1, \phi_2)(x^i) = 0$ (using $C^\infty X$-linearity of Hess in the $\phi_1, \phi_2$ variables). Thus we have

$$\sum_{i,j} (g_{ij} d_2 d_1 x^i d_2 d_1 x^j) = \langle F, F\rangle.$$

If we choose Riemann normal coordinates on $X$, the terms $d_1 g_{ij} d_2 d_1 x^i d_2 x^j$ and $d_2 g_{ij} d_1 x^i d_2 d_1 x^j$ vanish at the origin of $\{x^i\}$ for all $S$-points $\Phi$. In these coordinates, the second derivative of $g$ in the final term is the curvature of the metric. This shows

$$\mathcal{S}(\Phi) = \frac{1}{2}\langle F, F\rangle + \frac{1}{2} R(\phi_1, \phi_2, \phi_1, \phi_2),$$

proving Lemma 1.21 when $h = 0$.

For $h \neq 0$, we compute

$$\Phi^* h = f(h) + \theta_1 \phi_1(h) + \theta_2 \phi_2(h) + \theta_1\theta_2 E(h),$$

so when we integrate

$$\int_{S\times\mathbb{R}^{0|2}/S}\Phi^*h[d\theta_1 d\theta_2]=E(h)=F(h)+\mathrm{Hess}(\phi_1,\phi_2)h=\langle F,\nabla h\rangle+\mathrm{Hess}(\phi_1,\phi_2)h.$$

This computation together with the above shows

$$\begin{aligned}\mathcal{S}_\lambda(\Phi) &= \int_{S\times\mathbb{R}^{0|2}/S}\left(\frac{1}{2}\|T\Phi\|^2-\lambda(\Phi^*h)\mathrm{Ber}_{\mathbb{R}^{0|2}}\right)\\ &= \frac{1}{2}\langle F,F\rangle+\frac{1}{2}R(\phi_1,\phi_2,\phi_1,\phi_2)-\lambda\langle F,\nabla h\rangle-\lambda\,\mathrm{Hess}(\phi_1,\phi_2)h,\end{aligned}$$

which concludes the proof of Lemma 1.21.

*Remark* 3.4. This formula is in agreement with the "finesse" utilized in [DEF+99, Fre99, Fre01] to obtain the 1|2 supersymmetric quantum mechanics action functional from the Lagrangian density. For reference, the kinetic term in this action has the form

$$S(\gamma,\phi_1,\phi_2,F)=\int_\gamma\left(\frac{1}{2}\|\dot\gamma\|^2+\frac{1}{2}\langle\nabla_{\dot\gamma}\phi_1,\phi_1\rangle+\frac{1}{2}\langle\nabla_{\dot\gamma}\phi_2,\phi_2\rangle+\frac{1}{2}R(\phi_1,\phi_2,\phi_1,\phi_2)+\frac{1}{2}\|F\|^2\right)d\gamma$$

where $\gamma$ is a path in $X$, $\phi_i\in\Gamma(\gamma^*\pi TX)$ and $F\in\Gamma(\gamma^*TX)$. The dimensional reduction from 1|2 to 0|2 has the effect of only considering the constant paths in $X$ for which $\dot\gamma=0$. This recovers the action function we derived through supergeometric computations above.

3.4. **The Quantization.** In this section, we show how the action functional described above gives us a good notion of pushforward for 0|2-EFTs, proving Theorem 1.18. We define a map

$$\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2},X\times Y))\stackrel{\langle-\rangle^g}{\longrightarrow}\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2},Y)),$$

and show that it restricts to a map on field theories. Consider

$$\begin{aligned}\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2},X\times Y)) &\stackrel{\cdot e^{-\mathcal{S}}\otimes id}{\longrightarrow} C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|2},X))\otimes\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2},Y))\\ &\cong \Omega^\bullet(TX)\otimes\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2},Y))\\ &\stackrel{\text{project}\times\text{id}}{\longrightarrow}\Omega^{\text{top}}(TX)\otimes\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2},Y))\\ &\stackrel{\frac{1}{N}\int(-)\otimes id}{\longrightarrow}\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2},Y)),\end{aligned}$$

where the second line uses Proposition 2.19, and $\mathcal{S}$ is the action of the 0|2 sigma model on $X$. What remains is to check convergence of this integral. However, by how we've set things up, the image in $\Omega^\bullet(TX)$ consists of functions with polynomial growth in the noncompact direction, as discussed in 2.2.2. We claim that the Gaussian measure $e^{-\mathcal{S}}$ allows us to integrate all functions in the image.

It suffices to work locally to verify this claim, and furthermore we can set $Y=\mathrm{pt}$ for this preliminary part. So let $U\subset(\mathbb{R}^n,g)$ be a (bounded) open submanifold with coordinate $\{x^i\}$. Then $\underline{\mathsf{SM}}(\mathbb{R}^{0|2},U)\cong U\times\mathbb{R}^n\times R^{0|2n}$. Functions at a point $\Phi=(x,\phi_1,\phi_2,F)$ will be of the form

$$G(\Phi)=g(x)P(F)\omega(\phi_1)\eta(\phi_2)\in C^\infty(\underline{\mathsf{SM}}(\mathbb{R}^{0|2},U))$$

for $P$, $\omega$ and $\eta$ polynomials in $n$-variables. First we multiply by $e^{-\mathcal{S}}$, so we have

$$G(\Phi)e^{-\mathcal{S}(\Phi)}=e^{-F^2}e^{-R(\phi_1,\phi_2,\phi_1,\phi_2)}g(x)P(F)\omega(\phi_1)\eta(\phi_2)\tag{11}$$

Expanding this in in coordinates, we project to the coefficient of $\Pi_{i=1}^n d_1x^id_2x^i$, which we identify with the a section of the line bundle $\Omega^{\text{top}}(TU)$. The only problem we might encounter in convergence of the integral is in the $F$-variable. But $P(F)e^{-F^2}$ is integrable on $\mathbb{R}^n$, which completes the local argument.

Now we need to show that this map respects the action by $\underline{\mathsf{Euc}}(\mathbb{R}^{0|2})$, and for this we can no longer set $Y=\mathrm{pt}$. First we note that

$$0|2\text{-}\mathsf{EFT}^\bullet(X\times Y)\cong\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2},X\times Y))^{SO(2)\ltimes\mathbb{R}^{0|2}}.$$

There is a $\mathbb{Z}/2$-action left from the $O(2)$ action on $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X\times Y)$, and the $\pm 1$-eigenspaces of this action give $0|2\text{-}\mathsf{EFT}^{\mathrm{ev}}(X\times Y)$ and $0|2\text{-}\mathsf{EFT}^{\mathrm{odd}}(X\times Y)$ respectively. Packaged this way, the desired result follows from a few lemmas.

**Lemma 3.5.** *$\langle - \rangle$ restricts to a map on $SO(2)$-invariant functions.*

*Proof.* We claim the diagram commutes:

$$\begin{array}{ccc} \mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X\times Y))^{SO(2)} & \xrightarrow{\langle - \rangle} & \mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, Y)) \\ \text{proj}\downarrow & \nearrow & \\ \mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X))^{SO(2)}\otimes\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, Y))^{SO(2)} & & \end{array} \tag{12}$$

from which the Lemma will follow. So suppose that $\omega\otimes\eta\in\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X\times Y))$ is invariant under $SO(2)$. Then we have

$$\omega\otimes\eta = \sum_{k\in\mathbb{Z}}^{\text{finite}} \omega_k\otimes\eta_{-k}$$

where the subscript record the action of $SO(2)$ on the given element. Invariance of $\omega\otimes\eta$ requires that these representations pair up as indicated in the sum. Note that $e^{-\mathcal{S}}$ is $SO(2)$-invariant, so the quantity $(\omega\otimes\eta)\cdot(e^{-\mathcal{S}}\otimes 1)$ has the same decomposition as above. Working locally in coordinates $\{x^i\}$ on $X$, the integral will pick out the component of $d_1x^1d_2x^1\cdots d_1x^nd_2x^n$, and this is $SO(2)$-invariant. Hence,

$$\langle\omega\otimes\eta\rangle = \sum\langle\omega_k\otimes\eta_{-k}\rangle = \langle\omega_0\otimes\eta_0\rangle$$

proving the above claim. □

**Lemma 3.6.** *$\langle - \rangle$ restricts to a map on $SO(2)\rtimes\mathbb{R}^{0|2}$-invariant functions.*

*Proof.* Without loss of generality, suppose that $\omega\otimes\eta\in\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X))^{SO(2)}\otimes\mathcal{O}(\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, Y))^{SO(2)}$ and that $\omega\otimes\eta$ is $\mathbb{R}^{0|2}$-invariant. Therefore,

$$0 = d_i(\omega\otimes\eta) = d_i\omega\otimes\eta\pm\omega\otimes d_i\eta.$$

Now, the element $-\mathrm{id}\in SO(2)$ acts by the parity operator on the supermanifolds $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)$ and $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, Y)$, so $\omega$ and $\eta$ are even functions. Thus, $d_i\omega$ and $d_i\eta$ are odd functions on different supermanifolds and the only way for the above expression to vanish is for

$$d_i\omega = d_i\eta = 0.$$

Thus, by the definition of the integral, $\langle\omega\otimes\eta\rangle$ is $d_i$ invariant. By the previous lemma it is also $SO(2)$-invariant. □

**Proposition 3.7.** *$\langle - \rangle$ defines a map $0|2\text{-}\mathsf{EFT}^{\bullet}(X\times Y)\to 0|2\text{-}\mathsf{EFT}^{\bullet-n}(X\times Y)$ where $n$ is the dimension of $X$.*

*Proof.* It remains to understand the $\mathbb{Z}/2$-action on $SO(2)$-invariant functions arising from the $O(2)$-action. We note that $e^{-\mathcal{S}}$ is $O(2)$ invariant, so does not affect the degree of the map. Locally the integral first projects to $\Omega = d_1x^1d_2x^1\cdots d_1x^nd_2x^n$, which has degree $\dim(X)$ mod 2 with respect to the $O(2)$-action. Thus the affect of the integration lowers the degree by $n$. However, we also need to consider the degree of the coefficient of $\Omega$. This is some function on $TX$ whose degree is determined by the degree of the polynomial in the tangent direction. However, odd polynomials will integrate to zero (by applying Wick's lemma to equation 11) and so only the even polynomials contribute. These even polynomials have degree zero. Hence the total effect of quantizing lowers the degree of some $\omega\otimes\eta$ by $\deg(\Omega)$, and so $\langle\omega\otimes\eta\rangle\in 0|2\text{-}\mathsf{EFT}^{\bullet}(Y)$ differs from the degree of $\omega\otimes\eta\in 0|2\text{-}\mathsf{EFT}^{\bullet}(X\times Y)$ by $\dim(X)$ mod 2. □

Lastly, we fix the normalization constant $N = (2\pi)^{n/2}$, which completes the definition of the quantization.

## 4. The Chern-Gauss-Bonnet Theorem

The heart of the proof of the Chern-Gauss-Bonnet theorem was already presented in Section 1.6. Here we address the technical issues in the integration,

$$\pi_!^{q_\lambda}(1) = Z_X^\lambda = \int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2},X)} e^{-\mathcal{S}_\lambda(\Phi)}\frac{\mathcal{D}\Phi}{N},$$

and then give some remarks on how the computation allows one to interpret the Chern-Gauss-Bonnet formula as supersymmetric localization of a Feynman integral.

4.1. **Evaluating the Integrals.** First we compute the integral with $\lambda = 0$, and then compute it in the limit $\lambda \to \infty$.

In the definition of the integral in Section 3.4, first we project. We claim this picks out the correct part of $e^{-R(\phi_1,\phi_2,\phi_1,\phi_2)/2}$ to obtain the Pfaffian. Explicitly, we compute in coordinates from section 3.3

$$d_2 d_1 g_{ij} dx^i dx^j = R_{klij} d_1 x^k d_2 x^l d_1 x^i d_2 x^j$$

so if $\int(-)\mathcal{D}\Phi$ denotes the projection to the component of $d_1x^1 d_2x^2 \cdots d_1x^n d_2x^n$,

$$\int e^{-R(\phi_1,\phi_2,\phi_1,\phi_2)/2}\mathcal{D}\Phi = \frac{(-1)^{n/2}}{(n/2)!2^{n/2}} \sum \epsilon_{i_1\cdots i_n} R_{i_1 i_2 i_1 i_2} \cdots R_{i_{n-1}i_n i_{n-1} i_n} = \mathrm{Pf}(R).$$

Next, using Lemma 1.20, we identify $\Phi = (x, \phi_1, \phi_2, F)$ and first integrate out the $F$ variable. This is an ordinary Gaussian integral

$$\int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2},X)} e^{-\mathcal{S}}\frac{\mathcal{D}\Phi}{N} = \frac{1}{N}\int_X \mathrm{Pf}(R)\int_{TX/X} e^{-F^2/2}dF = \int_X \mathrm{Pf}(R).$$

So, recalling that $N = (2\pi)^{n/2}$, we get one side of the Chern-Gauss-Bonnet formula.

It remains to compute the integral in the $\lambda \to \infty$ limit. To employ the a localization argument, we first would like to integrate out the $F$-variable first

$$\int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2},X)} e^{-\mathcal{S}}\frac{\mathcal{D}\Phi}{N} = \frac{1}{N}\int_{\pi TX\oplus\pi TX}\Bigg(\exp\left(-R(\phi_1,\phi_2,\phi_1,\phi_2)/2 + \lambda\,\mathrm{Hess}(h)(\phi_1,\phi_2)\right) \cdot \int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2},X)/\pi TX\oplus\pi TX} \exp\left(-\langle F,F\rangle/2 - \lambda\langle\nabla h, F\rangle\right)\Bigg)$$

so we compute the Gaussian integral

$$\int_{\mathbb{R}^n} e^{-\langle F,F\rangle/2-\lambda\langle\nabla h,F\rangle}dF = (2\pi)^{n/2}e^{-\frac{\lambda^2}{2}\|\nabla h\|^2}.$$

Now we employ an argument similar to that of Mathai-Quillen [MQ86]. First, we assume that $h$ is Morse, and choose small disjoint open neighborhoods $U_p$ of the critical points of $h$. Let $X^c := X - \bigcup U_p$. Then

$$\int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2},X)} e^{-\mathcal{S}}\frac{\mathcal{D}\Phi}{N} = \int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2},X^c)} e^{-\mathcal{S}}\frac{\mathcal{D}\Phi}{N} + \int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2},\bigcup U_p)} e^{-\mathcal{S}}\frac{\mathcal{D}\Phi}{N}$$

We know that $\|\nabla h\|^2 > 0$ has a lower bound on $X^c$, so as $\lambda \to \infty$ we find

$$\int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2},X^c)} e^{-\mathcal{S}}\frac{\mathcal{D}\Phi}{N} = \int_{\pi TX^c\oplus\pi TX^c} \exp\left(-\frac{\lambda^2}{2}||\nabla h||^2 + \lambda\,\mathrm{Hess}(h)(\phi_1,\phi_2) - R(\phi_1,\phi_2,\phi_1,\phi_2)\right) \to 0.$$

it remains to evaluate the integral near the critical points of $h$. Focusing our attention on one such point $p$ (and possibly shrinking $U_p$), we choose coordinates on $U_p$ and via a concordance deform the metric to the standard one on $\mathbb{R}^n$. Since this metric is flat, $R = 0$. By arguments in Theorem 1.18, this concordance does not affect the value of the integral. We get the simplifed form

$$\int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2},U_p)} e^{-\mathcal{S}}\frac{\mathcal{D}\Phi}{N} = \int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2},U_p)} \exp\left(-\frac{\lambda^2}{2}\|\nabla h\|^2 + \lambda\,\mathrm{Hess}(h)(\phi_1,\phi_2)\right)$$

This integration amounts to a pair of Gaussian integrals. The odd integral is a fermionic Gaussian integral (see Appendix A) with respect to the pairing Hess($h$). So we find

$$\int_{\pi TU_p \oplus \pi TU_p} \exp(\lambda \, \mathrm{Hess}(h)(\phi_1, \phi_2)) \mathcal{D}\Phi = \lambda^n \det(\mathrm{Hess}(h)),$$

where the right hand side is understood to be a top-form on $U_p$. Hence,

$$\int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, U_p)} e^{-\mathcal{S}} \mathcal{D}\Phi = \lambda^n \int_{U_p} \exp\left(-\frac{\lambda^2}{2} ||\nabla h||^2\right) \det(\mathrm{Hess}(h)),$$

There are coordinates $\{x^i\}$ where the vector field $\nabla h$ on $U_p$ can be represented by a matrix $H_p$, where we get a vector field on $\mathbb{R}^n$ by $x \mapsto H_p x$. Note that in these coordinates, $\mathrm{Hess}(h) = H_p$ is symmetric and nondegenerate. We can repackage the above as

$$\int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, U_p)} e^{-\mathcal{S}} \frac{\mathcal{D}\Phi}{N} = \lambda^n \mathrm{sgn}(\det H_p) \int_{U_p} \exp\left(-\frac{\lambda^2}{2} \|H_p x\|^2\right) |\det(H_p)| dx^1 \wedge \cdots \wedge x^n.$$

This is another Gaussian integral, and in the limit $\lambda \to \infty$ the value of the integral on $U_p$ approaches the value of the integral on $\mathbb{R}^n$, which is

$$\int_{\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, U_p)} e^{-\mathcal{S}} \frac{\mathcal{D}\Phi}{N} = (2\pi)^{n/2} \mathrm{sgn}(\det \mathrm{Hess}(h)).$$

Summing over critical points

$$Z_X^\infty = (2\pi)^{n/2} \sum_{p \in \mathrm{zero}(dh)} \mathrm{sgn}(\det \mathrm{Hess}(h)) = (2\pi)^{n/2} \chi(X).$$

Comparing with $Z_X^0$ proves the Chern-Gauss-Bonnet theorem.

4.2. **Less Generic $h$.** By the contractibility of the space of functions, $Z_X = \langle 1 \rangle$ is the same for any function $h$. Using the arguments above, in the $\lambda \to \infty$ limit the integral computing $Z_X$ is supported on a small neighborhood of the critical locus inside $\underline{\mathsf{SM}}(\mathbb{R}^{0|2}, X)$. Thus, regardless of how degenerate a function $h$ is, a formal neighborhood of its critical locus contains the information of the Euler characteristic of the manifold. For example, if $h$ is Morse-Bott, we can use the Gauss-Codazzi equations in the usual manner to show that the Euler characteristic of $X$ is a sum over (signed) Euler characteristics of submanifolds.

4.3. **An Example of Localization.** The above computation can be seen as a toy model for localization in field theories with 2 supersymmetries. By this we mean that an integral over all fields turned out to depend only on a signed sum over the classical solutions, i.e., the critical points of $h$. In higher dimensions, one should expect signed information from other special classical solutions. For example, a signed count of gradient flow lines allows one to recover the de Rham cohomology of a manifold. The gradient flow lines are special classical solutions, namely, instantons.

*Remark* 4.1. As alluded to in the introduction, perhaps a more standard physical interpretation of the result in this paper is that we can understand $0|2$-field theories as the static configurations in supersymmetric mechanics, and so the minima of $h$ give the vacua of the quantum theory. As is well-known in the physics literature, we can calculate the $1|2$ partition function by a signed count of these vacua.

## Appendix A. Berezinian Integrals by Example

For a throughout treatment of (relative) Berezinian integrals, see the article by Deligne and Morgan [DEF$^+$99]. Here we wish to just give a few examples.

**Example A.1** (Relative integration)**.** We wish to explain relative integration for the trivial bundle $\mathbb{R}^{0|n} \times S \to S$. For any other family $M \to S$, locally the relative integration can be reduced to this example. On this bundle, the relative Berezinian is an $\mathcal{O}_{\mathbb{R}^{n|m}}$-module of rank $1|0$ if $m$ is even, and rank $0|1$ if $n$ is odd. A choice of coordinates $\theta_1, \ldots, \theta_n$ induces a trivialization of this module, and we denote trivializing section by $[d\theta_1 \cdots d\theta_n]$. If we tensor the relative Berezinian with the relative orientation bundle, we get a map

$$\int_{\mathbb{R}^{n|m}\times S/S} : C^\infty(\mathbb{R}^{n|m} \times S) \to C^\infty(S).$$

Consider first the case where $n = 0$. To Evaluate on a function, we Taylor expand in $\theta_i$ and project to the component of $\theta_1 \cdots \theta_n$, obtaining a function on $S$. When $n \neq 0$, first we project, obtaining a function on $C^\infty(\mathbb{R}^n \otimes S)$. Next we use an orientation form on $\mathbb{R}^n$ to integrate down to $C^\infty S$.

**Example A.2** (Fermionic Gaussians)**.** The following is standard but can be found, for example, in [GS]. Let $q$ be a quadratic form on a purely odd supervector space $V$ of even dimension. Note that for vectors $\omega, \eta \in V$, super quadratic means

$$q(\omega, \eta) = -q(\eta, \omega).$$

Thinking of functions on $V$ as being an exterior algebra, we see $q$ is in the 2nd antisymmetric power, so in particular, $q \in C^\infty(V)$. We claim

$$\int_V \exp\left(-\frac{1}{2}\tilde{q}\right) = \det(q)^{1/2}.$$

To see this, choose coordinates $C^\infty(V) \cong \mathbb{R}[\theta_1, \dots, \theta_{2n}]$ such that $q$ is a skew matrix of the form

$$q = \begin{pmatrix} \lambda_1 J & 0 & \dots & 0 \\ 0 & \lambda_2 J & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

The Berezinian integral projects to the top component of $\exp(-q/2)$, which is

$$(-2)^n \frac{1}{n!} q^n = \lambda_1 \cdots \lambda_n \theta_1 \cdots \theta_n.$$

Thus, the value of the integral is the product of the $\lambda_i$, whereas the determinant of $q$ is

$$\det(q) = \lambda_1^2 \cdots \lambda_n^2,$$

which verifies the claim.

*Remark* A.3. Compare the above with the usual Gaussian integral,

$$(2\pi)^{m/2} \int_W \exp\left(-\frac{1}{2}\tilde{q}\right) = \frac{1}{\det(q)^{1/2}}.$$

for $W$ an even (i.e., bosonic) $m$-dimensional vector space.

## Appendix B. Supermanifold Miscellany

B.1. **Functions on (Generalized) Supermanifolds.** One can identify the algebra of functions on a supermanifold $M$ with the superalgebra-valued functor $\mathsf{SM}(M, \mathbb{R}^{1|1})$ where addition and multiplication are defined using structures on the target $\mathbb{R}^{1|1}$. The grading on this algebra comes the the involution $\alpha$ of $\mathbb{R}^{1|1}$ determined by

$$\alpha^* : C^\infty(\mathbb{R})[\theta] \to C^\infty(\mathbb{R})[\theta], \quad \theta \mapsto -\theta$$

Using Yoneda, the morphisms $\mathsf{SM}(M, \mathbb{R}^{1|1})$ are determined by natural transformations between the functors $\underline{M}$ and $\underline{\mathbb{R}}^{1|1}$. Such a natural transformation is a map of sets

$$\mathsf{SM}(S, M) \to \mathsf{SM}(S, \mathbb{R}^{1|1}) \cong C^\infty(S).$$

Hence, maps of sets $\underline{M}(S) \to C^\infty(S)$ natural in $S$ are in bijection with functions on $M$. This notation makes sense for $M$ a generalized supermanifold. Being a functor valued in algebras, $C^\infty S$, generalized supermanifolds have an algebra of functions. In fact, since objects in the category of generalized manifolds can be written as a finite colimit of supermanifolds and countable colimits of nuclear vector spaces are nuclear, one can talk about the nuclear super vector space of functions on a generalized supermanifold (this fact was explained to me by Dmitri Pavlov).

B.2. **Extra Gradings.** Here we wish to explain a few features of the category $\mathsf{grSM}$ as compared to the more familiar $\mathsf{SM}$. First, we observe that $\mathsf{SM}$ includes as a full subcategory of $\mathsf{grSM}$, where a $n|m$ supermanifold becomes a $(n,0)|m$-supermanifold.

**Proposition B.1.** *There is a functor* $\mathsf{i}\colon \mathsf{grSM} \to \mathsf{SM}$.

*Remark* B.2. One can think of this as the $C^\infty$-completion of the polynomial functions on the given $(n,k)|m$-supermanifold.

*Proof.* For a $(n,k)|m$-supermanifold, we can can restrict its functor of points to the full subcategory $\mathsf{SM}$. We claim that this yields a representable supermanifold. To check this claim, it suffices to work locally. Additionally, this supermanifold is determined by its algebra of functions. By definition, the functions on the $(n,k)|m$ supermanifold look locally like $C^\infty(\mathbb{R}^n)\otimes\mathrm{Sym}^\bullet(\mathbb{R}^{k|m})$, and we denote the corresponding space by $\mathbb{R}^{(n,k)|m}$. Now, maps

$$\underline{\mathbb{R}}^{(n,k)|m}(S) \to C^\infty(S)$$

natural in $S$ (for $S$ an ordinary supermanifold) are in bijection with functions on $\mathbb{R}^{n+k|m}$. Indeed, we can form any smooth function of the polynomial coordinates on $\mathbb{R}^{(n,k)|m}$, and this algebra is in bijection with the smooth functions on $\mathbb{R}^{n+k|m}$. This proves the proposition. □

The following facts were explained to me by Dmitry Roytenberg. We recall them here for the readers convenience.

**Definition B.3.** A graded manifold is a manifold $|X|$ together with a sheaf $\mathcal{O}_X$ of $\mathbb{N}$-graded $\mathbb{R}$-algebras. A morphism of graded manifolds is a morphism of locally ringed spaces.

**Lemma B.4.** *Suppose that the monoid* $\mathbb{R}$ *acts smoothly on a (super) manifold* $X$*. Then the eigenvalues of the induced action on* $C^\infty X$ *are positive integers.*

*Proof.* Given an $\mathbb{R}^\times$-action on $C^\infty(X)$, the action on a given eigenspace is $r^\lambda$ for $r\in\mathbb{R}$ acting and $\lambda\in\mathbb{R}$ fixed. If this action extends smoothly to zero, this requires that $\lambda$ be non-negative. Let $k$ be large enough so that $\lambda - k$ is negative. Then the $k$th derivative of the action is differentiable if and only if $\lambda$ is an integer. This proves the lemma. □

**Proposition B.5.** *There is an equivalence of categories between* $\mathbb{N}$*-graded supermanifolds and supermanifolds with an action of the monoid* $\mathbb{R}$ *where morphisms are equivariant maps.*

*Sketch of Proof.* The functor $\mathsf{i}\colon\mathsf{grSM}\to\mathsf{SM}$ was explained in Proposition B.1. The inverse comes from the above lemma, where we take a direct sum over the eigenspaces of the $\mathbb{R}$-action on $C^\infty(M)$, obtaining an $\mathbb{N}$-graded algebra. We think of this graded algebra as a sheaf on the manifold $X$ whose functions are $C^\infty(M)/I$ where $I$ is the ideal generated by functions with non-zero grade (or, equivalently, the non-zero eigenspaces of the $\mathbb{R}$-action). Working locally on $X$ shows that the compositions of these functors give an equivalence at the level of objects. Since $\mathbb{R}$-equivariant maps will preserve eigenspaces, we also get an equivalence at the level of morphisms. □

*Remark* B.6. Claims related to the above Proposition can be found in [Sev01]. Partial results along these lines are proved in [GR11].

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