CONFORMAL FIELD THEORY SEMINAR NOTES

DANIEL BERWICK EVANS

Contents

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1. On Segal's "Definition of a CFT," Peter Teichner

In 1985 Segal wrote an amazing paper on CFTs. It's quite a long paper, so we won't be trying to understand every detail. Our goals are

(1) Understand examples of CFTs (free boson, free fermion, etc.)
(2) Translate Segal CFTs in VOA language (chapter 9 of the Segal notes)
(3) Today we'll give a survey of the first four chapters of Segal's notes (ending with the definition of a CFT).

First we'd like to know why we're studying 2-dimensional CFTs. The reason is there are some physical examples we'd like to understand, for example the 2-dimensional sigma model. The classical theory has as fields

$$\Phi_\Sigma := \mathsf{MAPS}(\Sigma, X)$$

where Σ is spacetime (a Riemannian or Lorentzian manifold) and the target manifold X is a Riemannian manifold. The classical action is a function on the space of fields, $S\colon \Phi \to \mathbb{R}$, which is given by

$$S(\phi) = \int_\Sigma ||T\phi||^2 dvol_\Sigma.$$

This functions comes from the operator norm on maps between normed vector spaces $T_p\phi\colon T_p\Sigma \to T_{\phi(p)}X$. (We could also add to the above action a potential function, B-field, etc. in the above action. For now we won't.)

Remark 1.1. The action $S(\phi)$ only depends on the conformal structure on Σ if and only if $d = 2$. In dimension 1 there is analogous result: the length of a path is parametrization invariant. However, this integral isn't related to the physicist notion of kinetic energy.

As usual, the classical solutions are the extrema of the action. For the following discussion, we'll assume that X is compact.

1.1. The 1-Dimensional Sigma Model: Classical and Quantum Mechanics on Manifolds.

When Σ is 1-dimensional we'd be studying the worldline of a point particle moving in a space X. The classical solutions are precisely the geodesics in X, parametrized by arc length. Note that this requirement on parametrization means that only the isometries of $\mathbb{R}$ (i.e. translations) act on classical solutions, not all diffeomorphisms of $\mathbb{R}$. As a space, the classical solutions are TX.

If we wish to "quantize" the above system, we'd be doing quantum mechanics of a point particle in X. We want a Hilbert space, $L^2(X)$, and a unitary operator that implements time evolution, $U_t\colon L^2X \to L^2X$. The operator kernel is given by a path integral,

$$U_t(x, y) = \int_{\phi\colon [0,t]\to X} e^{iS(\phi)/\hbar}\mathcal{D}\phi = e^{i\Delta t}$$

where the integral is over paths with boundary conditions $\phi(0) = x$ and $\phi(t) = y$ for points $x, y \in X$, and Δ is the Laplacian on X.

We might also wish to "quantize" observables. A classical observable is just a function on the classical solutions, $\mathcal{O} \in C^\infty(TX)$. In general, this is really tricky. If it happens that $\mathcal{O}$ is constant on the fibers (that is, comes from a function on X), then we get another operator kernel

$$\widehat{\mathcal{O}}(x, y) = \int_{\phi\colon [0,t]\to X} (f \circ \phi)e^{iS(\phi)/\hbar}\mathcal{D}\phi.$$

We might also be able to quantize functions that are linear in the fiber direction (basically a vector gets sent to its derivation on functions) but quantizing even the quadratic functions can be very hard.

1.2. The 2−Dimensional Sigma Model.

If with think of $\Sigma = S^1 \times \mathbb{R}$, then we can use the adjunction,

$$\mathsf{MAPS}(S^1 \times \mathbb{R}, X) \cong \mathsf{MAPS}(\mathbb{R}, LX),$$

so in some sense in dimension 2 we're doing classical and quantum mechanics on loop space. The classical solutions are harmonic maps of the cylinder into X, which are something like geodesics on LX. The space of classical solutions is TLX. The conformal maps of the cylinder act on these classical solutions. We'll call this group $\mathsf{Conf}(S^1 \times \mathbb{R})$.

We'd like to quantize in a similar fashion to the 1-dimensional case. Now, $L^2(LX)$ is defined (though this takes considerable effort) and we'd like to compute the integral

$$\text{“}U_t(x,y) = \int_{\phi\colon [0,t]\to LX} e^{iS(\phi)/\hbar}\mathcal{D}\phi\text{”}$$

where $\phi(0) = x$ and $\phi(1) = y$ are certain bounding circles $x, y \in LX$. Unfortunately, the above integral is not well-defined mathematically. The hope is that by some trickery we can define $U_{S^1\times[0,t]} = U_t$. In fact, the classical action is defined for any Riemann surface Σ, and we'd like a U_Σ for any such surface. In both cases, all the problems come down to defining a measure.

Remark 1.2. Although the integrand above (i.e. the classical action) is conformally invariant, the integral itself might not be. One way to say this is that the mysterious measure may not be conformally invariant.

In fact, the prediction from physics is that U_Σ are only conformally invariant when the target is sufficiently nice, e.g. Ricci flat.

We should talk a little bit about the symmetry from “time translation.” When $d = 1$, we found that the isometries of $\mathbb{R}$ act on the classical solutions, and when we quantized the isometries of $\mathbb{R}$ also acted by time translation by $e^{it\Delta}$. There is also a symmetry that arises from the path integral that comes from composing the U_t. It happens when $d = 1$ that these are the same symmetry, but now with $d = 2$ these are different. On the one hand, we expect symmetries of the classical solutions to act on the quantum solutions, namely we expect an action of $\mathsf{Conf}(S^1 \times \mathbb{R})$. On the other hand, the path integral gives a 1 parameter family of unitary operators that act on the Hilbert space.

Everything we're doing here is going to be with a Riemannian metric on Σ, but the physicists typically work with the Minkowski metric. In this setting there is a short exact sequence

$$\mathbb{Z} \to \mathsf{Conf}(S^1\times\mathbb{R}) \to \mathsf{Diff}^+(S^1_L) \times \mathsf{Diff}^+(S^1_R)$$

where the two groups at the end are of “left movers” and “right movers.” To explain this a little, there are two foliations of $\mathbb{R}^2$ by light-like paths (by “left movers” and “right movers”). When we curl up the spacelike direction, the leaves of this foliation will be circles. A conformal map will send light-like paths to light-like paths, so will give two diffeomorphisms of S^1, as mentioned above. The kernel of the map is given by an integer translations of $S^1 \times \mathbb{R}$.

Project 1: Prove that the above is an exact sequence.

BREAK

Definition 1.3. A Riemann surface Σ^2 is a compact smooth 2-manifold with an (almost) complex structure $J\colon T\Sigma \to T\Sigma$ with $J^2 = -1$. If Σ has boundary, J is also defined there. Furthermore, we want a choice of parametrization of the boundary, which is a diffeomorphism

$$\coprod S^1 \cong \partial\Sigma.$$

We call $\partial_{in}\Sigma$ the part of $\partial\Sigma$ where the diffeomorphism preserves orientation and $\partial_{out}\Sigma$ is the part of $\partial\Sigma$ the reverses the orientation.

The isomorphism

$$SO(2)\times\mathbb{R}_{>0} \cong \mathbb{C}^\times \cong Gl_1(\mathbb{C} \leq Gl_2(\mathbb{R})$$

show that the data in the first part of the definition can be formulated as a surface with conformal structure and volume form.

Project 2: Prove that the following is a symmetric monoidal category (denoted $2-\mathsf{CB}$). The objects are $\mathbb{N}_0$, which we think of as a finite collection of circles. The morphisms are given by isomorphism classes of Riemann surfaces (based on the above definition). The symmetric monoidal structure is disjoint union. Composition of morphisms is given by gluing Riemann surfaces along their boundary. That composition is well-defined comes “conformal welding.” (Notice that this category doesn't have any units. There are various remedies for this.)

Remark 1.4. If we wanted to build this bordism category for Riemannian manifolds (or some other geometry), the objects wouldn't be as simple: for a fixed circle we'd need some collar of geometry to glue things together. In general the moduli space of such circles will be huge. Somehow 2-dimensional conformal geometry makes this story much, much easier: the moduli space is just a single point.

Eventually, we will define a field theory to be a (projective) symmetric monoidal functor from the bordism category to VECT.

Why is this functorial perspective at all relevant when studying field theories? The simplest way to answer this is that the formal properties of the path integral "look like" a symmetric monoidal functor. Composition in the bordism category should give a composition of operators,

$$U_{\Sigma_1} \circ U_{\Sigma_2} = U_{\Sigma_1 \circ \Sigma_2}.$$

This tells us the functors might be a good thing to study. To see the monoidal structure entering, we consider the relation between "two string" systems and "one string" systems. So consider the Hilbert space of the theories:

$$H_{S^1} \cong L^2(S^1, X),$$

and

$$H_{S^1 \sqcup S^1} \cong L^2(C^\infty(S^1, X) \times C^\infty(S^1, X)) \cong H_{S^1} \otimes H_{S^1}.$$

Furthermore

$$H_\emptyset \cong L^2(C^\infty(\emptyset, X)) \cong \mathbb{C}.$$

This demonstrates the symmetric monoidal structure at the level of objects. Again by considering the path integral we expect

$$U_{\Sigma_1 \sqcup \Sigma_2} \cong U_{\Sigma_1} \otimes U_{\Sigma_2}$$

which gives the symmetric monoidal structure on morphisms. Since path integrals are hard, we'd be happy to find any theory that satisfies the above properties. Hence, we'll study these functors.

There is a particularly important semigroup,

$$\mathcal{A} := 2 - \mathsf{CB}(S^1, S^1),$$

the semigroup of annuli.

Project 3. Prove the following theorem.

Theorem 1.5. (1) *$\mathcal{A}$ is a complex manifold diffeomorphic to*

$$(0,1) \times \mathsf{Diff}^+(S^1_L) \times \mathsf{Diff}^+(S^1_R)/\Delta(SO(2)).$$

(2) *A semigroup structure is given by gluing the annuli together along boundaries and the gluing map, $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is holomorphic.*

(3) $T_A\mathcal{A} \cong \mathsf{Vect}_\mathbb{C}(S^1) \times \mathsf{Vect}_\mathbb{C}(S^1)/\mathsf{Vect}_{hol}(A)$

If we were to ignore parametrizations of the boundary, the moduli space of annuli is just $(0, 1)$, the conformal modulus of the annulus. By uniformization, this gives a standard annulus in the plane: its boundary is a circle of radius $r \in (0, 1)$ and a circle of radius 1. The if we add in the parametrizations we add two copies of $\mathsf{Diff}(S^1)$ to the moduli space, one for each component of the boundary. Finally, we need to take isomorphism classes. These are given by rigid rotations of the annulus, $SO(2)$.

We can make this all quite explicit by looking at some power series vanishing at 0 and ∞ on $\mathbb{P}^1$ and then finding annuli as subsets of $\mathbb{C}$. Peter did this and the picture is great, but it's also (rather directly) in Segal, so I won't include it here.

When we restrict a field theory to the semigroup of annuli, we expect to get a representation. In some sense, this is the main data of a field theory.

Theorem 1.6.

$$\{\text{Projective holomorphic reps of } \mathcal{A}\} \cong \{\text{Projective positive energy reps of } \mathsf{Diff}^+(S^1)\}$$

(where here we're talking about "essential equivalence classes" as in Presley-Segal.)

Recall that for a Lie group G,

$$\{\text{holomorphic reps of } G_\mathbb{C}\} \cong \{\text{reps of compact } G\}$$

so in this way $\mathcal{A}$ is a sort of complexification of $\mathsf{Diff}^+(S^1)$. But $\mathcal{A}$ isn't a group, so this analogy isn't totally precise.

There are nice subsets of each side of the bijection in the above theorem. We can say something about them too:

Theorem 1.7.

$$\{\text{reflection positive } U_A^* = U_{\bar{A}}\} \cong \{\text{unitary reps of } \mathsf{Diff}^+(S^1)\}$$

Here we can think of the bar operation, $\Sigma \mapsto \bar{\Sigma}$, on $2-\mathsf{EB}$ as interchanging the in and out boundaries, so the above theorem is saying that $U_{\bar{A}} = (U_A)^{-1} = U_{A^*}$ where the last equality uses reflection positivity. This is the definition of unitary.

There is a baby version of the above theorem that says that representations of S^1 are in bijection with holomorphic representations of $C^\times$, which are in bijection with semigroup representations of $D^\circ - \{0\}$.

2. Peter, Part II

Recall that $\mathcal{A}$ is a complex semigroup of annuli. This is the genus zero part of the bordism category of morphisms from the circle to itself:

$$\mathsf{CB}_{g=0}(S^1, S^1) \cong \mathcal{A}.$$

We can parametrize $\mathcal{A}$ by picking two power series that converge on the disk,

$$f_0 = a_1 z + a_2 z^2 + \dots,\ a_1 \neq 0 \quad f_1 = z^{-1} + b_2 z^{-2} + \dots$$

where $f_0|_{\partial D}$ is the incoming parametrization of the boundary of the annulus, and $f_1|_{\partial D}$ is the outgoing and we require that these two curves do not intersect. Note that by setting $b_1 = 1$ we've rescaled the metric on $\mathbb{C}$, so this description has already taken the quotient by conformal isomorphisms.

From this we see that $\mathcal{A} \subseteq Hol_0(D^2) \times Hol_\infty(D^2)$ and this is an open subset. Each of these factors is a $\mathbb{C}$ Frechet space (in fact, a $\mathbb{C}$-nuclear space), so in this sense we find that $\mathcal{A}$ is an infinite dimensional complex manifold.

Lemma 2.1. *The semigroup multiplication map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is holomorphic.*

Furthermore, we find the tangent space at an annulus A to be

$$T_A\mathcal{A} \cong \mathsf{Vect}_{\mathbb{C}}(S^1) \times \mathsf{Vect}_{\mathbb{C}}(S^1)/\mathsf{Vect}_{Hol}(A),$$

where the first two factors "push" the boundary circles in $\mathbb{C}$ around.

Here's a brief argument about conformal wielding. See if you buy it.

We'd like to glue Σ to Σ' along their boundaries. Both of these surfaces have (almost) complex structures, J and J'. We also have a diffeomorphism, $\phi\colon \partial\Sigma \to \partial\Sigma'$. So pick a nonvanishing vector field C along $\partial\Sigma$. Then the flow of JC defines a collar on Σ. We can do the same thing on $\partial\Sigma'$. Then the diffeomorphism ϕ extends to a diffeomorphism of collars. Also, we see that $J = J'$ in the gluing. Together this gives a conformal structure on $\Sigma \sqcup_\phi \Sigma'$. All that remains to show is that the construction is independent of the choice of C, but any conformal map will rescale both C and the collar, giving rise to a conformally isomorphic surface.

Remark 2.2. A CFT gives a *projective* representation of $\mathcal{A}$.

Recall from last time

Theorem 2.3. *Essential equivalence classes of*

$$\{\text{Projective holomorphic reps of } \mathcal{A}\} \cong \{\text{Projective positive energy reps of } \mathsf{Diff}^+(S^1)\}$$

We can think of the subgroup of "standard annuli" where $a_1 = q$ and $a_i = 0$ for $i > 0$ and $b_i = 0$. This is some subset of the left hand side. On the right hand side we have $S^1 \subset \mathsf{Diff}^+(S^1)$, and in some sense this equivalence is saying that holomorphic maps on the boundary of the disk extend across the interior. In another sense, the complexification of S^1 is $\mathbb{C}^\times$, and then the standard annuli sit inside of $\mathbb{C}^\times$. So from this perspective, representations of a compact Lie group are in bijection with holomorphic representations of the complexification, and then this gives a representation of D°, which we consider as a subsemigroup of $\mathcal{A}$.

Theorem 2.4. *Essential equivalence classes of*

$$\{\text{holomorphic extensions of } \mathcal{A} \text{ by } \mathbb{C}^\times\} \cong \{\text{smooth extensions of } \mathsf{Diff}^+(S^1) \text{ by } \mathbb{C}^\times\}$$

The right hand side is determined by a pair

$$(c,h) \in \mathbb{C} \times \mathbb{C}^{\times} \cong \mathsf{Hom}(\mathbb{R},\mathbb{C}) \times \mathsf{Hom}(\mathbb{Z},\mathbb{C}^{\times})$$

where c is called the central charge. We can read of h from the extension

$$\begin{array}{ccccc} \mathbb{Z} & \to & \mathbb{R} & \to & SO(2) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \to & \mathsf{Diff}^+(\mathbb{R}^1) & \to & \mathsf{Diff}^+(S^1) \end{array} \tag{1}$$

and $1 \in \mathbb{Z} \mapsto h \in \mathbb{R}$.

The central charge c is defined by

$$\begin{array}{ccccc} \mathbb{C}^{\times} & \to & \mathsf{Diff}^+(S^1) & \to & \mathsf{Diff}^+(S^1) \\ \downarrow & & \downarrow & & \downarrow \rho \\ \mathbb{C}^{\times} & \to & GL(\mathcal{H}) & \to & PGL(\mathcal{H}) \end{array} \tag{2}$$

where the right hand square is a pullback square. Then $c(\rho) \in \mathbb{C}$ is determined by the Lie algebra representation:

$$[\rho(L_{-n}), \rho(L_n)] = -2in\rho(L_0) + c(\rho)\frac{n(n^2-1)}{12} \cdot \mathrm{id}_{\mathcal{H}},$$

where $L_n = e^{inx}\partial/\partial x$ and note that $[L_{-n}, L_n] = -2inL_0$.

2.1. **Segal's Definition of a CFT.** A CFT is given by the following data

(1) A projective representation $\mathcal{H}$ of $\mathsf{Diff}^+(S^1)$;
(2) For each Riemann surface Σ with $\coprod_n S^1 \cong \partial_{in}\Sigma$ and $\coprod_m S^1 \cong \partial_{out}\Sigma$, a line of trace-class operators $U_\Sigma \colon \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes m}$ that depends smoothly on the conformal structure on Σ such that for a gluing $\Sigma \sqcup_\phi \Sigma'$

$$U_\Sigma \circ U_{\Sigma'} = U_{\Sigma \sqcup_\phi \Sigma'}$$

and

$$U_{\Sigma \sqcup \Sigma'} = U_\Sigma \otimes U_{\Sigma'}.$$

A CFT is *rational* if

$$\mathcal{H} = \bigoplus_i^{\text{finite}} \mathcal{H}_L(i) \otimes \mathcal{H}_R(i)$$

where the $\mathcal{H}_L$ and $\mathcal{H}_R$ are weak holomorphic and antiholomophic CFTs (respectively).

A *weak CFT* (basically) is the same as a CFT, except that rather than a line of trace-class operators one has a finite dimensional subspace, called a conformal block.

Remark 2.5. Weak holomorphic CFTs "are" fusion categories, and hence "are" (3-2-1 extended) 3d TFTs. For example, WZW as a weak 2d CFT corresponds to Chern-Simons theory as a 3-2-1 TFT. $Z(S^1)$ is how one gets the fusion category. Hopefully we will make this more precise in a few weeks.

The FFRS theorem says that picking a symmetric, special Frobenius algebra object in $Z(S^1)$ leads to a rational CFT. They use special fusion categories coming from vertex operator algebras to show this.
Projects: Explain the last few paragraphs.

2.2. **The Determinant Line.** An example of the translation from a weak holomorphic CFT to a 3d TFT is given by the determinant line.

So, unravelling definitions a 3d TFT must assign a finite dimensional vector space to a closed surface. Let's say we have a weak CFT. Then for a closed surface Σ we have $E \to \mathcal{J}(\Sigma)$, a vector bundle over the moduli stack of conformal structures on Σ. This vector bundle comes equipped with a projectively flat connection, the Quillen connection. We can make this flat by tensoring with a power of the determinant line,

$$E \otimes det(\Sigma)^{\otimes c}$$

where c is the central charge. The flat sections of this resulting bundle gives the finite dimensional vector space that the CFT associated to a closed surface.

Remark 2.6. WE NEED TO BE A LITTLE CAREFUL HERE BECAUSE THE CLOSED SURFACE GIVES A MAP FROM $\mathbb{C} \to \mathbb{C}$. We need to have a more precise definition of the conformal blocks.

This $det(\Sigma)^{\otimes c}$ makes sense for "rigged" surfaces. This gives a central extension

$$\mathbb{Z} \to \mathsf{Aut}(\Sigma, \text{rigging}) \to \mathsf{Aut}(\Sigma)$$

and so this gives the (familiar?) projective anomalies for theories like Chern-Simons.

So let's describe this determinant line.

Given Σ a Riemann surface, consider $\bar{\partial}\colon \Omega^0(\Sigma, \mathbb{C}) \to \Omega^{0,1}(\Sigma, \mathbb{C})$ with boundary conditions given by

$$C^\infty_{\geq 0}(S^1, \mathbb{C}),$$

the functions on the circle extending to holomorphic functions on the disk (the "positive Fourier modes"). This gives an elliptic (in particular, Fredholm) operator. So the determinant line is defined:

$$\det(\bar{\partial}_\Sigma) = \Lambda^{\text{top}}\ker(\bar{\partial}_\Sigma) \otimes \Lambda^{\text{top}}\text{coker}(\bar{\partial}_\Sigma)^*.$$

These fit into a holomorphic line bundle on $\mathcal{J}(\Sigma)$, where

$$\mathcal{M}_\Sigma = \mathcal{J}(\Sigma)/\mathsf{Diff}^+(\Sigma).$$

We can see that the diffeomorphisms act nontrivially on the line by considering the torus $\mathbb{R}^2/\mathbb{Z}^2$. We can rotate the square by $\pi/4$, and we find that this multiplies the determinant line by i. This gives a line bundle on the moduli *stack* but not the moduli *space*.

From this we find the (rigged) diffeomorphisms of Σ act on these sections, as we'd expect in a TFT.

Remark 2.7. We can restrict this bundle to annuli. For annuli, the conformal blocks are one dimensional. One can show that compositions in the semigroup of annuli will have tensor products of these determinant lines over them. This gives a semigroup extension, where $(c, h) = (-2, 0)$. This "shows" that $h = 0$ always for CFTs.

More generally on surfaces we have a canonical isomorphism

$$\text{“}\det(\Sigma \sqcup_\phi \Sigma') \cong \det(\Sigma') \otimes \det(\Sigma),\text{''}$$

where really this isomorphism takes place by pulling back the determinant line on $\Sigma \sqcup_\phi \Sigma'$ via the conformal wielding map,

$$\mathsf{Diff}(\partial\Sigma, \partial\Sigma') \times \mathcal{J}(\Sigma) \times \mathcal{J}(\Sigma') \to \mathcal{J}(\Sigma \sqcup \Sigma'),$$

where there is also a canonical determinant line on $\mathsf{Diff}(\partial\Sigma, \partial\Sigma') \cong \coprod^n \mathsf{Diff}^+(S^1)$.

3. Modular Functors, Weakly Conformal Field Theories, and Twisted Field Theories, Dmitri Pavlov

Today we want to explain some aspects of Segal's definition of a weakly conformal field theory, and put these ideas into a more modern context, namely twisted field theories. A modular functor in Segal's world gives rise to a weakly conformal field theory. In the Stolz-Teichner language a modular functor is a type of twist, and a weakly conformal field theory is a twisted field theory.

Outline:

(1) Modular functors and weakly conformal field theories
(2) Twists and twisted CFTs
(3) Loop groups and CFTs
(4) Determinant line and the free boson

3.1. Segal's Chapter 5.

Definition 3.1. Let Φ be a finite set. Define the symmetric monoidal stack S_Φ as having objects (families of) Riemann surfaces X with parametrized boundary ∂X, where each connected component of ∂X is marked by an element of Φ. A morphism $X \to Y$ is a smooth surjective immersion that is an isomorphism on $\text{Int}(X) = X - \partial X$, and on the boundary we can glue together pairs of circles (where one circle is incoming and one outgoing) marked by the same elements of Φ.

For example, self-gluings of surfaces are morphisms, as are diffeomorphisms rel boundary. In fact, we can have diffeomorphisms that permute the boundary components, but they must leave the parametrizations fixed.

(Note that this is not a stack fibered in groupoids, the the terminology is slightly nonstandard. However, this fibered category does satisfy descent, so is stacky in some sense.)

Definition 3.2. Let Φ be a finite set. A modular functor is a strong symmetric monoidal functor

$$E\colon S_\Phi \to \mathsf{Vect}_{\mathbb{C}},$$

such that

(1) For all objects Σ in S_Φ, let G_Σ be the set of all objects that are the same as Σ but with different labels on some fixed set of pairs of boundary circles $A \subset \partial X$. Both circles in each pair must have the same label, hence the cardinality of G_Σ is $\#\Phi^{\#(\text{pairs in } A)}$. We require the following map to be an isomorphism:

$$\bigoplus_{\Sigma' \in G_\Sigma} E(\Sigma') \to E(S)$$

where S is gotten from Σ by gluing pairs of circles in A.

Basically, if we glue Σ to itself and get a surface S, we want to sum over all ways to cut S to get Σ and label the resulting circles, and then glue back together again and get S.

(2) There is an isomorphism $E(S^2) \cong \mathbb{C}$.

We also require that for every $\phi \in \Phi$, there is at least one surface where E does not vanish when it is labeled by ϕ.

In the above "strong" monoidal means the canonical map

$$E(X) \otimes E(Y) \to E(X \sqcup Y)$$

is an isomorphism.

A consequence of this is that for an annulus $A_{\phi,\psi}$ with parametrized boundary labeled by ϕ and ψ, $E(A) = 0$ when $\phi \neq \psi$ and $E(A) = \mathbb{C}$ when $\phi = \psi$. To see this, build a matrix with entries $a_{\phi\psi} = \dim(E(A_{\phi,\psi}))$. Furthermore, $a^2 = a$. This means a is a projector. The above nonvanishing requirement forces this matrix to be the identity, and the result follows.

We also get a unique element $1 \in \Phi$. We see this by chopping the sphere into two discs with parmetrized boundary ϕ, so

$$\mathbb{C} = E(S^2) = \bigoplus_{\phi \in \Phi} E(D^2_\phi) \otimes E({}_\phi D^2),$$

so all of these factors are zero except for a single one, and this gives our unique $1 \in \Phi$.

Definition 3.3. Let Φ be a finite set and E a modular functor. Then define a new category $\mathsf{C^E}$ where

$$\mathsf{Obj}(\mathsf{C^E}) = \coprod_{\mathsf{k} \geq 0} \mathsf{\Phi^k}$$

$$\mathsf{Mor}(A, B) = (X, \partial X = A \sqcup B, \epsilon \in E(X))$$

where X is a Riemann surface and $A \sqcup B$ labels the boundary. Composition of the Riemann surfaces is the usual story. Composing $\epsilon \in E(X)$ and $\epsilon' \in E(X')$, we have a canonical map

$$E(X) \otimes E(X') \to E(X \circ Y)$$

given to us by the strong monoidal structure together with with the gluing map $X \sqcup X' \to X \circ X'$.

Remark 3.4. The category doesn't have identities. We can add in "thin bordisms" where are diffeomorphisms of the circle to fix this issue, but we won't. At least for now. (In particular, dealing with the labelings looks like a minor headache.)

Definition 3.5. A weakly conformal field theory is a symmetric monoidal functor $\mathsf{C^E} \to \mathsf{TVS}$, where the target is topological vector spaces with the projective tensor product.

The point here is that for each Riemann surface X, our CFT functor gives a finite dimensional vector space of maps (called the conformal block). If for all X, $E(X)$ is one dimensional, we get a line bundle on the moduli space of Riemann surfaces, and we return to the first definition of a CFT (and in particular, the functor is projective).

3.2. **Modernizing the Language.** Let C be the bicategory whose objects are finite collections of circles, morphisms are Riemann surfaces and 2-morphisms are isomorphisms of Riemann surfaces.

Definition 3.6. A twist is a symmetric monoidal functor $T\colon \mathsf{C} \to \mathsf{vNA}$ where the target category is the bicategory of von Neumann algebras, bimodules and intertwiners (though in these examples, all algebras will be finite dimensional).

To construct one of these from a modular functor (Φ, E), on objects we need only specify it for the generator of objects, namely a single circle. So

$$T(S^1) = \bigoplus_{\phi\in\Phi} \mathbb{C}$$

Then $T(X)$ is a $T(X_{\text{in}})$-$T(X_{\text{out}})$-bimodule, and each of these algebras are just sums over as many $\mathbb{C}$'s as there are labelings. To give such a bimodule we just need to assign a vector space for every labeling of the boundary of X. So let A be a labeling of X_{in}; then we assign the vector space $E(X_A)$.

For an isomorphism of Riemann surfaces $h\colon X \to X$, we get an arrow $E(X_A) \to E(X_A)$ for every A, which gives the associated (invertible) intertwiner.

Then property (1) in the definition of modular functor shows that gluing surfaces corresponds to tensor product of bimodules, rather that the canonical arrow

$$T(X) \otimes_{T(V)} T(X') \to T(X \circ X')$$

is an isomorphism, where

$$U \xrightarrow{X} V \xrightarrow{X'} W.$$

Definition 3.7. Let T be a twist. A CFT twisted by T is a symmetric monoidal natural transformation $1 \to T$, where 1 is the constant functor to the symmetric monoidal unit of vNA.

More carefully, $1(S^1) = \mathbb{C}$, $1(X) = \mathbb{C}$ as a $\mathbb{C}$-$\mathbb{C}$ bimodule, and to isomorphisms of Riemann surfaces we assign the identity map $\mathbb{C} \to \mathbb{C}$.

Remark 3.8. When $T = 1$, we get the "usual" notion of a CFT. A little bit of diagram chasing shows this.

4. Dmitri, Part II

Recall from last time: For Φ a finite set we have a category S_Φ whose objects are Riemann surfaces with parametrized boundary labeled by Φ and morphisms are sewings of surfaces along boundaries. From this we can define a modular functor, $E\colon S_\Phi \to \mathsf{Vect}_\mathbb{C}$, which is a strong monoidal functor such that

(1) For all objects Σ of S_Φ, and all cuttings f of Σ along disjoint embedded circles, the following arrow is an isomorphism

$$\bigoplus_f E(f)\colon \bigoplus_f E(\mathrm{dom}(f)) \xrightarrow{\cong} E(\Sigma)$$

where $\mathrm{dom}(f)$ is a labeling of the cut surface.

(2) $E(S^2) = \mathbb{C}$.

Remark 4.1. The first condition is some sort of locality axiom, and will be reinterpretted in a moment: A twist of a field theory must respect compositions.

Then we defined the category C^E whose objects are parametrized circles labeled by Φ and morphisms are Riemann surfaces Σ together with $e \in E(\Sigma)$. Composition makes sense because a modular functor was a *strong* monoidal functor. Then a weakly conformal CFT is a functor $H\colon \mathsf{C}^\mathsf{E} \to \mathsf{TVS}_\mathbb{C}$. We think that implicit in this definition is the linear dependence of the functor on $e \in E$.

Although they display a tremendous amount of insight into field theories, these definitions are relatively awkward. We wish to use the more modern language of twisted field theories to understand what is going on.

A twist is a symmetric monoidal functor $T\colon \mathsf{C} \to \mathsf{ALG}^\times$, where the target is the category of algebras, bimodules and (invertible) intertwiners. Let us now recall how to construct a twist from a modular functor.

Let $A \in \mathsf{C}$ be an object. Then

$$T(A) = \bigoplus_\phi \mathbb{C}$$

where ϕ is a labeling of A, and there are $\Phi^{\#A}$ of these. On morphisms $\Sigma \in \mathsf{C}(\mathsf{A}, \mathsf{B})$, define

$$_{T(A)}T(\Sigma)_{T(B)} = \bigoplus_{\phi \in \mathrm{lab}(A),\ \psi \in \mathrm{lab}(B)} E(_\phi X_\psi).$$

In a bicategory, composition is extra data, so now we need to write it down, and we'll use condition (1) from the definition of a modular functor. So if we have $_A X_B$ and $_B Y_C$, then we have

$$\bigoplus_{\chi \in \mathrm{lab}(B)} E(_\phi X_\chi) \otimes E(_\chi Y_\psi) \to E(_\phi X \circ Y_\psi)$$

and by property (1) this is an isomorphism, so allows us to define T for compositions.

Remark 4.2. Note that these twists land in the subcategory 2-VECT of ALG, the category of 2-vector spaces: objects are "vectors of vector spaces" and the bimodules are "matrices of vector spaces."

Now recall that a field theory twisted by T is a natural transformation from 1 to T, where 1 and T are thought of as functors from $\mathsf{C} \to \mathsf{ALG}$.

We'd like to build a twisted field theory (T, U) from a weakly conformal one (Φ, E, H). So define

$$_{1(A)}U(A)_{T(A)} = {}_{\mathbb{C}}U(A)_{T(A)} := \bigoplus_{\phi \in \mathrm{lab}(A)} H(A_\phi).$$

(Notice again we're still in 2-VECT.) The interesting part is in defining $U(_A X_B)$. Consider the following commutative diagram:

$$\begin{array}{ccc} 1(A) & \xrightarrow{1(X)} & 1(B) \\ \downarrow{\scriptstyle U(A)} & \nearrow & \downarrow{\scriptstyle U(B)} \\ T(A) & \xrightarrow{T(X)} & T(B) \end{array}$$

The 2-morphism in the above diagram goes from $U(A) \otimes_{T(A)} T(X)$ to $1(X) \otimes_{1(B)} U(B) = U(B)$. Thus we want a $\mathbb{C}$-$T(B)$-bimodule morphism

$$U(A) \otimes_{T(A)} T(X)_{T(B)} \to U(B)_{T(B)}$$

and we'll specify this morphism for each ψ a labeling of B, and then sum over all labelings. Chasing some definitions, the arrow we need is

$$\bigoplus_{\phi \in \mathrm{lab}(A)} H(A_\phi) \otimes_{\mathbb{C}} E(_\phi X_\psi) \to H(B_\psi),$$

and such a morphism is determined by morphisms for each ϕ

$$H(A_\phi) \otimes_{\mathbb{C}} E(_\phi X_\psi) \to H(B_\psi)$$

where $v \otimes \epsilon \mapsto H(X, \epsilon)(v)$. Here we are using the linear dependence assumption mentioned above, but not present in Segal's original article. To spell this out, an element $\epsilon \in E(_\phi X_\psi)$ gives a map

$$H(_\phi X_\psi, \epsilon) \colon H(A_\phi) \to H(B_\psi)$$

and we want this to give a map (which uses the linearity assumption)

$$E(_\phi X_\psi) \to \underline{\mathsf{Hom}}(H(A_\phi), H(B_\psi)),$$

and by adjunction this in turn gives

$$H(A_\phi) \otimes_{\mathbb{C}} E(_\phi X_\psi) \to H(B_\psi).$$

Then after summing over ψ, ϕ, we get the map we were after.

Remark 4.3. All of this is only using the 2-VECT part of ALG, so somehow the representations of CB that we're using are really boring so far.

4.1. **Examples.** Maybe someone can give a talk on this in a few weeks:

Theorem 4.4. *Integer powers of the determinant line give all modular functors with* $\#\Phi = 1$.

4.1.1. *The determinant line.* Let $\#\Phi = 1$ and $T(A) = \mathbb{C}$. Then let $T({}_AX_B) := \det(X)$, the determinant line on X. One way to define this is to let $Y = X \sqcup_{\partial X} \coprod D^2$ by a closed surface gotten from X after gluing in some discs. Then

$$\det(X) := \Lambda^{\text{top}}\Omega^1_{\text{hol}}(Y)^* = \Lambda^{\text{top}}(\ker(\bar{\partial})^*) \otimes \Lambda^{\text{top}}(\text{coker}(\bar{\partial}))$$

where

$$\bar{\partial}\colon \Omega^{0,0}(Y) \to \Omega^{0,1}(Y).$$

Gluing the disks to X is the same as choosing boundary conditions that make $\bar{\partial}$ elliptic so that the second expression makes sense. We can see the isomorphism above by noting

$$\bar{\partial}_0^* = \bar{\partial}_1\colon \Omega^{1,0}Y \to \Omega^{1,1}Y,$$

and

$$\bar{\partial}_0\colon \Omega^{0,0}Y \to \Omega^{0,1}Y$$

and then

$$\Lambda^{\text{top}}\Omega^1_{\text{hol}}(Y)^* = \ker(\bar{\partial}_1) = \ker(\bar{\partial}_0^*) = \text{coker}(\bar{\partial}_0)$$

and $\ker(\bar{\partial}) = \mathbb{C}$ canonically.

Remark 4.5. Seeing that this gives a holomorphic line bundle isn't so trivial; it's all in a paper by Quillen. To understand the holomorphic stuff functorially will require us to actually work in families, and our functors are fibered functors.

So we've defined the twists on objects and morphisms. We'd need to check compositions, but this gets into some somewhat nontrivial territory. See the appendix in Segal; all the relevant details can (probably) be found there.

Define $L_X := \text{image}(\text{res})$ where

$$\text{res}\colon \Omega^{0,0}_{\text{hol}}(X) \oplus \Omega^{0,1}_{\text{hol}}(X) \to C^\infty(\partial X, \mathbb{C}) \oplus \Omega^1(\partial X, \mathbb{C})$$

This L_X gives a polarization of $C^\infty(\partial X, \mathbb{C})$, which allows us to implement the Fock space construction. In particular it is a Lagrangian subspace with respect to the natural symplectic structure on $C^\infty(\partial X, \mathbb{C}) \oplus \Omega^1(\partial X, \mathbb{C})$ given by pairing an integration over the boundary. Furthermore, there is a Heisenburg group, $\text{Heis}(\partial X)$, and $\Lambda^\bullet(L_X^*)$ is an irreducible $\text{Heis}(\partial X)$-module.

Now we can define

$$U(A) := F_A \cong \Lambda^\bullet(L_A)$$

where F is for "Fock," and here $L_A := L_{\coprod D^2}$ where the disks are the boundaries of the circles in A. It remains to define U for surfaces. From the Fock space construction, there is an irreducible $\text{Heis}(A)$ action on L_A. First note that

$$T({}_AX_B) = \mathsf{Hom}_{\text{Heis}(A)}(F_A, F_X \otimes_{\text{Heis}(B)} F_B)$$

(which is not so obvious, but it is a theorem.) Now to define $U({}_AX_B)$ we require a map

$$U(A) \otimes_{T(A)} T(X) \to U(B)$$

which we give by $(x \otimes \phi) \mapsto (\phi(v \otimes \Omega_X))$ where Ω is the vacuum vector in the Fock space construction.

Remark 4.6. We have to be a little careful in claiming $\text{Heis}_{A\sqcup B} = \text{Heis}_A \otimes \text{Heis}_B$. There is a central extension of heis and a quotient that makes this work. I think it's something like

$$\text{heis}_{\partial X} := U(\text{Heis}_{\partial X})/\lambda \cdot 1 = \lambda \cdot c$$

where c is an element of the central extension defining $\text{Heis}_{\partial X}$. For this algebra we have

$$\text{heis}_A \otimes \text{heis}_B \cong \text{heis}_{A\sqcup B}.$$

Remark 4.7. To get out of 2-VECT we can assign $T(A) = \text{heis}_A$ and $T({}_AX_B) = F_X$. In this version it is also clear that $T(A)$ has a natural $\text{Diff}(A)$-action. If we were working in the internal category world, we would actually require this action from the beginning. Somehow here we've taken a "Morita equivalent" twist, which we can't do if we have the $\text{Diff}(A)$-action.

5. Peter, Free Field Theories

These theories are really boring physically, but give examples of functorial field theories. There are two reasons we'll consider this in detail. On the one hand, if free theories didn't give an example of a field theory, our definitions would be nonsense. On the other hand, functorial field theories are usually so hard to construct it is worthwhile to figure things out in this (relatively) simple situation.

Recall that a field theory in our way of thinking is a natural transformation $\mathsf{Nat}(1, T)$ where $T\colon d{-}\mathsf{Bord} \to \mathsf{ALG}$ is a twist functor and 1 is the constant functor to the symmetric monoidal unit.

There is a category Symp and a functor $\mathsf{Fock}\colon (\mathsf{Symp}, \oplus) \to (\mathsf{ALG}, \otimes)$. A free field theory is one that factors as a natural transformation $\mathsf{Fock} \circ 1 \to \mathsf{Fock} \circ T'$ where $T'\colon d - \mathsf{Bord} \to \mathsf{Symp}$ is some other kind of twist functor.

The way to think of this is that the functor Fock is a "second quantization" and the natural transformation $1 \to T'$ is a prequantization. Roughly the category Symp has as objects symplectic vector spaces and morphisms $V \to V'$ are Lagrangian subspace of $-V \oplus V'$. The 2-morphisms are (probably?) linear maps between Lagrangians. So more or less, by construction the space of fields of free theories is linear, and the functor Fock is geometric quantization on the symplectic manifold of classical solutions.

There are some tricky points in the category Symp that are reminiscent of the familiar issues in the Fukaya category. Namely, composition of 1-morphisms is hard to define on the nose. In the linear case these things should just work, but there are some details to check. We want for $V \xrightarrow{L_1} V' \xrightarrow{L_2} V''$ to define

$$L_{12} = L_1 \circ L_2 := \{(v_1, v_2) \mid \exists v \in V \text{ with } (v_1, v) \in L_1,\ (v, v_2) \in L_2\}.$$

So it is clear that the symplectic form vanishes on L_{12}, but it isn't clear that this subspace is "maximal." Normally one would just count dimensions, but in this infinite dimensional setting one must be more careful.

Now let's describe the functor Fock. We'll do this for $\mathbb{Z}/2$-graded V, anticipating applications to supersymmetric theories. Then on the odd part of the vector space,

$$\mathsf{Fock}(V^{\text{odd}}, \omega) := CL(V^{\text{odd}}, \omega) = \mathcal{U}(\mathbb{C} \to \cdots \to (V^{\text{odd}}, +))/\lambda c = \lambda 1$$

where the universal enveloping algebra is for some super Lie algebra. On the even part of the vector space,

$$\mathsf{Fock}(V^{\text{ev}}, \omega) := \text{Heis}(V, \omega) = \mathcal{U}(\mathbb{C} \to \text{heis} \to (V, +))/\lambda c = \lambda 1$$

where $\mathcal{U}$ denotes the universal enveloping algebra. We can also define these things in terms of some quotient of the tensor algebra of V, from which we see

$$\mathsf{Fock}(V, \omega) = \mathcal{U}(\mathbb{C} \to \cdots \to (V, +))/\lambda c = \lambda 1.$$

To save space, we'll refer to Fock applied to a vector space as being a clifford algebra, $\text{Cl}(V)$. In all of these, $c \in \mathbb{C}$ is in the original Lie algebra, and 1 is the unit of the universal enveloping algebra. So, the above define Fock on the objects of Symp. Notice that this functor is indeed monoidal: a sum of vector spaces will get mapped to a tensor product of algebras. Furthermore, $\text{Cl}(-V) = \text{Cl}(V)^{\text{op}}$.

To define Fock on 1-morphisms, we need to construct a $\text{Cl}(V)$-module for any $L \subseteq V$:

$$\mathsf{Fock}(L) := \text{Cl}(V) \otimes_{\text{Cl}(L)} \mathbb{C},$$

where $\text{Cl}(V)$ acts on $\mathbb{C}$ via an isomorphism $\text{Cl}(L) \cong \Lambda^\bullet(L) \xrightarrow{\epsilon} \mathbb{C}$. We have a vacuum vector $\Omega := 1 \otimes 1 \in \mathsf{Fock}(L)$. Notice that Ω is $\text{Cl}(V)$-cyclic: by acting on Ω by $\text{Cl}(V)$, we obtain all elements of $\mathsf{Fock}(L)$. Furthermore, Ω is annihilated by L (which can be deduced by considering the augmentation map ϵ). The complimentary Lagrangian to L acts by creators.

6. Peter, Examples of Free Field Theories

Last time we explained a bit about how to obtain a free theory from certain differential operators. We defined the Fock space functor on the kernel of these operators, and free field theories were defined as functors that factor in this way. Now, the problem is when one wants to quantize in families. Then the kernel of the Dirac operator doesn't vary nicely, so one needs to get the Fock space construction to work on something that does vary nicely. One approach would be to do this for the entire spectrum of the Dirac operator, but I wasn't able to figure this out in time.

Instead today we'll restrict our attention to a particular example. A lot of this is in "What is an Elliptic Object?"

First we need to understand the line bundle of densities over the world sheet, Σ^d. (We're probably going to assume at various points that $d = 1$ or 2). Let $L^k \to \Sigma^d$ be the real line bundle whose fiber at a point x is

$$L_x^k := \{\rho : \Lambda^d T_x\Sigma \to \mathbb{R} \mid \rho(\lambda \cdot \omega) = |\lambda|^{k/d} \cdot \rho(\omega)\}$$

for $k \in \mathbb{C}$. Note that these bundles are trivial, but not trivialized. They are canonically oriented, however. For a d manifold, we can integrate densities: $C^\infty(L^d) \xrightarrow{\int_\Sigma} \mathbb{R}$.

Lemma 6.1. *A conformal structure on Σ induces a canonical metric on the "weightless cotangent bundle:"*

$$T_0^*\Sigma := L^{-1} \otimes T^*\Sigma.$$

Sketch of Proof. This is a little calculation. Pick a metric in the conformal class, and pick a section of L^{-1}. Then this gives a metric on $T_0^*\Sigma$. It remains to show that varying the metric within it's conformal class doesn't change this metric. □

Definition 6.2. Let E^n be a real n-dimensional vector bundle over Σ, with a metric. A spin structure on E is a $\mathbb{Z}/2$-graded irreducible $Cl(E)$-Cl_n bimodule bundle S on Σ.

Remark 6.3. There are exactly two isomorphism classes of such bimodules on each fiber,

$$_{Cl(E_x)}(S_x)_{Cl_n} \cong_{Cl_n} (Cl_n)_{Cl_n},$$

and there are two of these Cl_n-bimodules, the one above and $(Cl_n)^{op}$ (which is just the original module, but with the reversed grading). We can see this using the fact that Cl_n is simple and has no proper ideals and maybe some other stuff. An orientation of E picks out one of these isomorphism classes, and the other way around, so an orientation of E specifies a spin structure uniquely (if one exists).

Now notice that there is a double covering

$$Spin((E_x, S_x), (E_y, S_y)) \to SO(E_x, E_y)$$

where we define

$$Spin((E_x, S_x), (E_y, S_y)) := \{(\phi, \tilde{\phi}) \mid \phi \in SO(E_x, E_y), \quad \tilde{\phi} :_{Cl(E_x)} (S_x)_{Cl_n} \xrightarrow{\cong}_{Cl(E_y)} (S_y)_{Cl_n}\}$$

where we note that since Cl is a functor we have $Cl(\phi) : Cl(E_x) \xrightarrow{\cong} Cl(E_y)$. A priori, the second isomorphism (by Schur's lemma) could be multiplication by any $\mathbb{R}^\times$, but because we require an isomorphism to preserve the metric, we can only multiply by ± 1, which shows that we get a 2-sheeted covering.

Remark 6.4. If V is a Cl_n-module, we can define the V-spinors as $S \times_{Cl_n} V \to \Sigma$, and this gives the usual notion of spinor bundles. The above approach is a little more systematic because we don't have to worry about certain mod 8 behaviors.

Definition 6.5. If Σ^d is conformal, a spin structure S on Σ is a spin structure on the weightless tangent bundle (coming from the canonical metric on this bundle).

Definition 6.6. Given a spin structure S on Σ, the Cl_n-linear Dirac operator is

$$D : C^\infty(\Sigma, L^{(d-1)/2} \otimes S \xrightarrow{\nabla_g} C^\infty(\Sigma, T^*\Sigma \otimes L^{(d-1)/2} \otimes S) = C^\infty(\Sigma, L^{(d+1)/2} \otimes T_0^*\Sigma \otimes S) \xrightarrow{mult} C^\infty(\Sigma, L^{(d+1)/2} \otimes S)$$

where ∇ is the Levi-Civita connection associated to the (choice of) metric g, and *mult* is Clifford multiplication,

$$T_0^*\Sigma \otimes S \to S.$$

We can define this for k-densities but with $k = (d-1)/2$ we claim that the map only depends on the conformal class, not on the choice of metric.

Remark 6.7. This is a natural pairing between k and $k+1$ forms if and only if $k = (d-1)/2$, so if we wish to say anything like "the Dirac operator is self-adjoint," we are also forced into this choice.

Now we wish to give Green's formula for this operator. So let $\phi, \psi \in C^\infty(\Sigma, L^{(d-1)/2} \otimes S)$. Then

$$\langle D\psi, \phi \rangle_\Sigma - \langle \psi, D\phi \rangle_\Sigma = \int_{\partial\Sigma} \langle c(\nu) \cdot \psi \mid_\partial, \phi \mid_\partial \rangle_{\partial\Sigma} dx.$$

To make sense out of this, we need to know how these fancy spin structures restrict to the boundary.

(The following is somewhat parenthetical. Skip to the remark 6.10 for a quicker way.)

Lemma 6.8. *There is a functor from the category of spin structures on $E \oplus \underline{\mathbb{R}}$ to the category of spin structures on E*

This is important since the restrict of $T\Sigma$ to the boundary is $T\partial\Sigma \oplus \underline{\mathbb{R}}$ where the $\underline{\mathbb{R}}$ direction is spanned by a normal vector. We need this to understand (for example) Clifford multiplication on the boundary.

Proof. The map is

$$_{Cl(E\oplus\underline{\mathbb{R}}}S_{Cl_{n+1}} \to_{Cl(E)} S^{ev}_{Cl_n}$$

where the map $Cl(E) \to Cl(E \oplus \underline{\mathbb{R}})$ is given by $v \mapsto e_1 \cdot v$ and Cl_n just includes into Cl_{n+1}. The grading on S^{ev} is given by $s \mapsto e_1^E s e_1$. It is clear this map squares to 1, and its eigenspaces give the grading.

We'd need to check that these maps respect all the Clifford relations. This is computational. □

Corollary 6.9. *A spin structure on Σ induces a spin structure on $\partial\Sigma$.*

In our case,

$$L_\Sigma^{-1} \otimes T^*\Sigma \mid_{\partial\Sigma} \cong (L_{\partial\Sigma}^{-1} \otimes T^*\partial\Sigma) \oplus \underline{\mathbb{R}}$$

with the isomorphism determined by a choice of normal vector.

Remark 6.10. If one can interpret ν as a section of the weightless cotangent space, the above formula takes place in the restriction bundle

$$C^\infty(\Sigma, L_\Sigma^{(d-1)/2} \otimes S) \stackrel{res_\partial}{\to} C^\infty(\partial\Sigma, (L_\Sigma^{(d-1)/2} \otimes S) \mid_{\partial\Sigma})$$

Note that the Dirac operator above is odd, since the embedding $E \otimes S \hookrightarrow Cl(E) \otimes S$ is in the sense that the action satisfies

$$E \otimes S^{ev} \to S^{odd}.$$

Now define the operator D^+ as a composition

$$C^\infty(L^{(d-1)/2} \otimes S^{ev}) \stackrel{D}{\to} C^\infty(L^{(d+1)/2} \otimes S^{odd}) \stackrel{\cdot e_1}{\to} C^\infty(L^{(d+1)/2} \otimes S^{ev}).$$

Let's see what the Green's formula looks like for D^+. So

$$\langle D^+\psi, \phi\rangle_\Sigma + \rangle\psi, D^+\phi\rangle = \rangle D\psi e_1, \phi\rangle + \rangle\psi, D\phi e_1\rangle = \int_{\partial\Sigma} \langle c(\nu) \cdot \psi \mid_\partial \cdot e_1, \phi \mid_\partial\rangle = \int_{\partial\Sigma} \langle \alpha(\psi \mid_\partial), \phi \mid_\partial\rangle$$

where α denotes the grading involution. In particular, this D^+ is now skew adjoint when $\partial\Sigma = \emptyset$.

7. Harold, Positive Energy Representations of Loops Groups and the Virasoro Algebra

Let G be a connected, simply connected compact Lie group. Consider

$$LG := C^\infty(S^1, G)$$

as a topological group (with the Fréchet topology). We want to study the representation theory of this group. From considerations in physics and field theories, we want certain representations, namely the finite energy ones. It will turn out that there aren't very many of these. However, there are a lot of interesting *projective* finite energy representations.

7.1. **Central Extensions of LG.** First we want to connect positive energy projective representations to central extensions of LG. So let $\langle -, - \rangle$ be an ad-invariant symmetric form on $\mathfrak{g}$ and $\xi, \eta \in L\mathfrak{g}$. Then set

$$\omega(\xi, \eta) = \int_{S^1} \langle \xi, d\eta \rangle.$$

By the Jacobi identity for $\mathfrak{g}$ and integration by parts, we see that this leads to a 2-cocycle, $H^2(L\mathfrak{g})$. Now set

$$\widehat{L\mathfrak{g}} := L\mathfrak{g} \oplus \mathbb{R}$$

which is a Lie algebra with bracket

$$[(\xi, a), (\eta, b)] = ([\xi, \eta), \omega(\xi, \eta)).$$

Any central extension of $L\mathfrak{g}$ arises in this way, since H^2 is 1-dimensional and we can rescale the cocycle ω. Note that all of these lead to isomorphic Lie algebras (given by rescaling ω), but non-isomorphic extensions.

Now we want to integrate this up to the Lie group. So we want to build

$$1 \to S^1 \to \widehat{LG} \to LG \to 1.$$

and we'll do this by considering ω as the curvature of a connection on the above circle bundle on LG. So choose a splitting

$$\widehat{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathrm{Lie}(S^1).$$

Then extend this splitting to $T\widehat{LG}$ by left translations. We want $S^1 \subset \widehat{LG}$ to be normal, so the above splitting leads to a connection α on $S^1 \to \widehat{LG} \to LG$.

Now if ξ, η are vector field on LG and $\hat{\xi}, \hat{\eta}$ are their lifts to $\widehat{LG}$, then $[\hat{\xi}, \hat{\eta}] - \widehat{[\xi, \eta]}$ identifies $L\mathfrak{g}$ 2-cocycles with the curvature of α (which is the Chern class of the S^1 bundle in question). Note that we get $\mathbb{Z}$-many of these, which picks out $\mathbb{Z} \subset \mathbb{R}$ where $\mathbb{R}$ classified the Lie algebra extensions.

Proposition 7.1. *Fix $\langle -, - \rangle$ so that $\langle \theta, \theta \rangle = 2$, where θ is the highest root. Then the corresponding $\omega \in H^2(L\mathfrak{g})$ generates the integral classes.*

Above, we've assumed that we had a group extension and then classified things assuming they exist. But given an invariant form on LG extending some cocycle ω we can construct a circle bundle $S^1 \to P \to LG$ with a connection. Then take $\widehat{LG}$ as the group of diffeomorphisms $P \to P$ that

(1) cover the left action of LG on itself and
(2) preserve the given connection.

Once we choose an identity element (which is in the fiber of the identity of LG) We can identify this group with the total space of the bundle since $\pi_2(G) = 0$ and hence LG is simply connected.

Remark 7.2. If we had chosen are loop group to be continuous maps S^1 to G, there are no extensions. So the story is perhaps more subtle than it appears.

Now fix $\widehat{LG}$ as the extension corresponding to $\langle \theta, \theta \rangle = 2$. This is the universal central extension of LG. We'll study this throughout. In fact, we'll be interested in

$$\tilde{LG} = \widehat{LG} \rtimes S^1_{rot}$$

where S^1_{rot} acts on LG by precomposition.

Remark 7.3. It looks like we can build a semidirect product with $\mathsf{Diff}^+(S^1)$ rather than just S^1_{rot}. However, the construction above of $\widehat{LG}$ wasn't completely canonical, so maybe that's why we don't get it.

7.2. Quick Review of Compact Lie Groups.

Let $T \hookrightarrow G$ be a maximal torus. Then define

$$\check{T} := \mathsf{Hom}(S^1, T)$$

$$\hat{T} := \mathsf{Hom}(T, S^1).$$

Taking derivatives, we embed

$$\check{T} \hookrightarrow \mathfrak{t}, \quad \hat{T} \hookrightarrow \mathfrak{t}^*$$

where $\mathfrak{t}$ is the Lie algebra of T. Now,

$$W := \frac{N_G(T)}{T},$$

is a finite group, and it acts on $\hat{T}$ and $\check{T}$. Given $\alpha \in \hat{T}$, let

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid t \cdot x = \alpha(t)x.$$

If $\mathfrak{g}_\alpha \neq 0$ and $\alpha \neq 0$, then α is called a *root*.

Remark 7.4. We need to be a little careful here with differences between $\mathfrak{g}$ and it's complexification; for example, we need to define the circle action on $\mathfrak{g}$ above and this happens in the complexification.

The *coroot* $h_\alpha \in \mathfrak{t}$ is determined by

(1) $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha]$
(2) $\alpha(h_\alpha) = 2$.

7.3. **The Affine Weyl Group.** Now we claim that a maximal torus of LG is given by

$$T_R \times T \times T_c \hookrightarrow \tilde{LG}$$

where T_R is the circle of rotations (acting by precomposition), T is the constant loops to $T \subset G$, and T_c is the central S^1 in the extension.

Proposition 7.5.

$$W_{\text{aff}} := \frac{N(T_R \times T \times T_c)}{T_R \times T \times T_c} = \check{T} \rtimes W.$$

Sketch of Proof. If $R_\theta \in T_R$ and $f \in LG$, then $R_\theta^{-1} f(z) R_\theta = f(e^{i\theta} z)$. Then if $f \in \check{T}$,

$$f(z)R_\theta f(z)^{-1} = R_\theta f(e^{i\theta}z)f(z)^{-1} = R_\theta f(e^{i\theta}zz^{-1}) \in T_R \times T.$$

□

Under the action of $\mathbb{T} = T_r \times T \times T_c$, $\tilde{L\mathfrak{g}}$ decomposes under the action of T_R (via Fourier modes) as

$$\mathbb{C}_R \oplus \left(\bigoplus_{k \in \mathbb{Z}} g z^k \right) \oplus \mathbb{C}_c$$

and the roots are

$$\{(k, \alpha, 0) \in \mathbb{C}_R^* \oplus \mathfrak{t}^* \oplus \mathbb{C}_c^* \mid k \in \mathbb{Z}, \quad \alpha = 0 \quad \text{or} \quad \alpha \text{ a root of } \mathfrak{g}\},$$

and by convention (or convenience), $(0, 0, 0)$ is not a root.

For example with $LG = LSU(2)$, we can draw some pictures of the root lattice.

The coroots in $\mathbb{C}_R \oplus \mathfrak{t} \oplus \mathbb{C}_c$ are

$$h_{(k,\alpha,0)} = (0, h_\alpha, \frac{-k}{2}||h_\alpha||^2).$$

Remark 7.6. The integral scalings of $\langle -, - \rangle$ are characterized by $||h_\alpha||^2$ being even for all roots α, so $k||h_\alpha||^2/2$ is an integer.

If V is a $\tilde{LG}$ representation we can write it as

$$\bigoplus_{\mathbb{Z}_R \times \hat{T} \times \mathbb{Z}_c} V_{n,\lambda,\ell}.$$

Definition 7.7. V has positive energy if $V_{n,\lambda,\ell} = 0$ for all $n < 0$.

Remark 7.8. We can motivate positive energy representations in two ways: on the one hand, from physics we want a vacuum vector and creation operators that fill out the space of states. On the other hand from representation theory, we want a highest (or lowest) weight vector. These two things are really the same mathematically, but philosophically give independent motivations for studying these guys.

Since T_c is central, when V is irreducible then T_c acts by a unique eigenvalue (the level of V).

Remark 7.9. We could have assumed that the energy is bounded from below (rather than being positive). By twisting such representations by T_R characters, we may assume the lowest nonzero energy level is 0.

Theorem 7.10. *Irreducible positive energy representations are in bijection with antidominant weights.*

Let's think about the action of the affine Weyl group. So $\eta \in \check{T}$ (which is the "affine part" of the affine Weyl group) acts by

$$\eta \cdot (n, \lambda, \ell) = (\eta + \lambda(\eta) + \frac{\ell}{2}||\eta||^2, \lambda + \ell\eta^*, \ell).$$

In particular it preserves the level.

The picture to have in mind for $LSU(2)$ is an integer lattice with a parabola. We think of point on the lattice nearest the vertex of the parabola as $(0, 0, \ell)$, and then from the above η acts by

$$(0, 0, \ell) \mapsto (\frac{\ell}{2}||\eta||^2, \ell\eta^*, \ell),$$

so we see a parabola in the first two variables. We fill in the parabola by looking the action of $SU(2)$ by constant loops, and perhaps some other tricky reasoning. (Note that the action by W is just reflection about the vertical axis, so doesn't tell us much.)

The simple positive roots of $\tilde{LG}$ are

$$\{(0,\alpha,0) \mid \alpha \text{ simple positive}\}$$

and $(1,-\theta,0)$ where θ is a highest root of $\mathfrak{g}$.

Definition 7.11. A weight ω is antidominatn if $\omega(h_\alpha) \le 0$ for all simple coroots h_α.

Notice that we can remove the word "simple," as the condition is equivalent.

The simple coroot corresponding to $(1,-\theta,0)$ is $(0,-h_\theta,\frac{-1}{2}||h_\theta||^2) = (0,-h_\theta,-1)$, where the equality follows from our normalization. So, $(0,\lambda,\ell)$ is antidominant if

(1) $(0,\lambda,\ell)\cdot(0,\alpha,0) \le 0$ for α simple which holds if and only if λ is antidominant for G.
(2) $(0,\lambda,\ell)\cdot(0,-h_\theta,-1) \le 0$, which is equivalent to saying $-\lambda(h_\theta)-\ell \le 0$.

Putting this all together gives a little triangle of possible antidominant weights at a fixed level, and hence a finite number of positive energy representations. This will be the input to the modular functor when we look to build a CFT from loop group representations.

8. Loop Groups to CFTs, Dmitri

First we'll outline the definitions. Next we'll see that for a fixed group and level we obtain a twist and a twisted CFT. Finally, we'll consider some of the modularity properties.

Recall the symmetric monoidal bicategory of conformal cobordisms, C:

(1) Objects: disjoint unions of standard circles (so in bijection with $\mathbb{N}$),
(2) 1-morphisms: Riemann surfaces with boundary, and
(3) 2-morphisms: conformal isomorphisms of Riemann surfaces rel boundary.

The target category, W^*, for our field theories has

(1) Objects: von Neumann algebras,
(2) 1-morphisms: bimodules,
(3) 2-morphisms: intertwiners.

Definition 8.1. A twist is a functor between these categories.

For example, we have the constant functor to the symmetric monoidal unit. Applied to objects if gives $\mathbb{C}$, to 1-morphisms $\mathbb{C}$ as a $\mathbb{C}$-$\mathbb{C}$ bimodule and to 2-morphisms we assign the identity intertwiner.

Definition 8.2. Let T be a twist. A T-twisted CFT U is a symmetric monoidal natural transformation $1 \to T$.

Let G be a complexification of a simply connected compact Lie group. Let $\ell \in H^4(BG) \cong \mathbb{Z}$ be the level. Define $\Phi_{G,\ell}$ as the (finite) set of isomorphism classes of positive energy irreducible representations of LG at level ℓ.

Given G,ℓ as above, we can construct a twist, $T_{G,\ell} : \mathsf{C} \to \mathsf{W}^*$ as follows.

(1) On objects: $T_{G,\ell}(A) = \bigoplus_{\Phi^{\pi_0 A}} \mathbb{C}$, so in particular $T(S^1) = \bigoplus_\Phi \mathbb{C}$.
(2) On 1-morphisms: $T({}_A\Sigma_B) := \mathsf{Hom}_{Hol(\Sigma,G)}(U(A),U(B))$, where

$$U(A) := \boxtimes_{\pi_0 A} \bigoplus_{\phi\in\Phi} \phi \in \mathsf{Rep}((\widetilde{LG}^{\ell})^{\pi_0 A}) = \mathsf{Map}^{\ell}(A,G)$$

where $\boxtimes$ is the tensor product is the external tensor product,

$$_A M_B \boxtimes_C N_D =_{A\otimes C} (M \boxtimes N)_{B\otimes D}$$

compared to

$$_A M_B \otimes_B {}_B N_C =_A (M\otimes_B N)_C.$$

Now we form the pullback

$$\begin{array}{ccccccc} U(A),U(B) \curvearrowleft Hol(\Sigma,G) & \overset{res}{\to} & \mathsf{Map}(\partial\Sigma,G) = (LG)^{\pi_0\partial\Sigma} & & \\ \uparrow\exists\downarrow & & \uparrow & & \\ \widetilde{Hol}^{\ell}(\Sigma,G) & \to & (\widetilde{LG^{\pi_0\partial\Sigma}})^{\ell} = \widetilde{\mathsf{Map}}^{\ell}(\partial\Sigma,G) & \rightrightarrows & (\widetilde{LG}^{\ell})^{\pi_0 A},\ (\widetilde{LG}^{\ell})^{\pi_0 B} \end{array}$$

Segal claims there is a *canonical* splitting where denoted above.

$$\begin{array}{ccccc}
(S^1)^{\pi_0 A} & \to & (\widetilde{LG}^{\ell})^{\pi_0 A} & \to & \mathsf{Map}(A,G) \\
\downarrow & & \downarrow & & \downarrow \\
S^1 & \to & (\widetilde{LG^{\pi_0 A}})^{\ell} & \to & \mathsf{Map}(A,G) \\
 & & \curvearrowleft & & \uparrow \\
 & & U(A) & \nwarrow & Hol(\Sigma, G)
\end{array}$$

Let's work this out in the case of $G = \mathbb{C}^\times$ (which isn't simply connected, but this issue won't trouble us). So $Hol(\Sigma, \mathbb{C}^\times) = Hol(D, \mathbb{C}^\times)$, and $\mathsf{Map}(A, \mathbb{C}^\times) = \mathsf{Map}(S^1, \mathbb{C}^\times)$. (Then there is some application of Stokes theorem that I didn't quite catch which gives the splitting. For general G, constructing this splitting still isn't entirely clear. Need to clean this part up.) Let f and g be holomorphic functions on the circle that extend to F and G which are holomorphic functions on Σ. Then

$$\int_{S^1} FdG = \int_\Sigma d(FdG) = \int_\Sigma dF \wedge dG = \int_\Sigma \partial F \wedge \partial G \in \Omega^{2,0}\Sigma = 0.$$

where $d = \partial + \overline{\partial}$.

(3) On 2-morphisms: $T(\phi) = id$. Though this makes some of us nervous since then the mapping class group doesn't act.

Now, we'd need to show that T preserves composition.

Now we need to define $U_{G,\ell}$, the T-twisted conformal field theory:

(1) On objects: $U(A)$ was defined above.
(2) On 1-morphisms: $U({}_A\Sigma_B)$ is given by

$$\begin{array}{ccc}
1(A) & \xrightarrow{U(A)} & 1(B) \\
\Big\downarrow{\scriptstyle U(A)} & \nearrow & \Big\downarrow{\scriptstyle U(B)} \\
T(A) & \xrightarrow{T(X)} & T(B)
\end{array}$$

(3) On 2-morphisms.... so tired of drawing diagrams...

Again we need to show that U preserves compositions, though this may not be so easy (and in fact might still be open).

Now that we've defined the functors, we need to check they satisfy the axioms. This is still open in general. The vertex operator algebra people get a full theory, but that is just the "genus 0" part of the Segal CFT.

We have partition functions

$$\chi_\phi(q, g) = \sum_k q^k \operatorname{tr}(g|_{\phi_k}),$$

where k is the energy and ϕ_k is the energy k subspace. Here g is a clutching function, and there are many details that are confusing.

9. AQFT, Ansgar

Let M^4, g be a Lorentzian (with signature $(+,-,-,-)$) manifold that is orientable and time-orientable.

Definition 9.1. A QFT on M is a functor $\mathsf{BOpen}(M) \to C^*\mathsf{ALG}^1$ where $\mathsf{BOpen}(M)$ is the category of bounded open subsets of M and $C^*{-}\mathsf{Alg}^1$ is the category whose objects are C^* algebras and whose morphisms are injective $*$-homomorphisms. This functor must satisfy:

(1) $A_{ql} := \operatorname{colim} A(\mathcal{O}) \in C^*$-alg has an irreducible faithful representation. (So the Hilbert space of states is not part of the data!)
(2) $[A(\mathcal{O}_1), A(\mathcal{O}_2)]_{A(\mathcal{O}_3)} = 0$ for $\mathcal{O}_1$ and $\mathcal{O}_2$ causally separated open subsets (meaning that there is no time-like curve that connects $\mathcal{O}_1$ and $\mathcal{O}_2$).
(3) $\mathcal{O}_1 \subset \mathcal{D}(\mathcal{O}_2) \implies A(\mathcal{O}_1) \subset A(\mathcal{O}_2)$ where $\mathcal{D}(\mathcal{O}_2)$ is the double cone containing the region $\mathcal{O}_2$.
(4) There exists an action $\mathsf{Isom}(M,g) \overset{\alpha}{\hookrightarrow} \mathsf{Aut}(A_{ql})$ so $x \in \mathsf{Isom}(M,g)$

$$\alpha_x(A(\mathcal{O})) \subset A(x\mathcal{O}).$$

We will think of (and often refer to) $A(\mathcal{O})$ as the algebra of observables localized in $\mathcal{O}$.

We will assume that (M, g) is globally hyperbolic, i.e. there exists a smooth spacelike codimension 1 submanifold that hits each inextendable timelike curve (and thus submanifold is called the Cauchy surface). It is a fact that such manifolds M are diffeomorphic to $\mathbb{R} \times \Sigma$, so (by our orientability assumptions) M is spin.

9.1. **Classical Theory.** Now, $Cl(1,3)$ has up to isomorphism a unique representation by 4×4 matrices acting on $\mathbb{C}^4$. These are the usually γ-matrices:

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}$$

where η_{ab} is the standard Lorentzian metric with signature $(+,-,-,-)$. So let

$$DM := SM \times_{\mathrm{Spin}} \mathbb{C}^4.$$

We have a Dirac operator

$$\eth : C^\infty(DM) \xrightarrow{\nabla} C^\infty(T^*M \otimes DM) \to C^\infty(TM \otimes DM) \xrightarrow{\gamma} C^\infty(DM)$$

where γ is Clifford multiplication.

From this we get the Dirac equation(s)

$$(-i\eth + m)u = 0, \quad u \in C^\infty(DM)$$
$$(i\eth + m)v = 0, \quad v \in C^\infty(D^*M)$$

We have a map $C^\infty(DM) \xrightarrow{\cong} C^\infty(DM)$ $u \mapsto u^\dagger$ that has the property that u satisfies the first Dirac equation above if and only if $u^\dagger$ satisfies the second one.

Theorem 9.2. *The exists a unique $S^\pm : C_0^\infty(DM) \to C^\infty(DM)$ such that:*

1. $(-i\eth + m)S^\pm = i = S^\pm(-i\eth + m)$ *where $i : C_0^\infty(DM) \hookrightarrow C^\infty(DM)$ is the inclusion of compactly supported sections into the space of all sections.*
2. $\mathrm{supp}(S^\pm f) \subseteq \mathcal{J}^\pm(\mathrm{supp}(f))$.

Note that solutions to the Dirac equation cannot have compact support; we will force them to have compact support in the space direction, but then they have noncompact support in the time direction.

We define

$$S := S^+ - S^-,$$

and call this the *propogator.*

Theorem 9.3. *The restriction map is an isomorphism:*

$$\{\text{classical solutions with spacelike compact support}\} \xrightarrow{\text{res}} C_0^\infty(D\Sigma).$$

with inverse given by

$$C_0^\infty(D\Sigma) \xrightarrow{-i\nu\cdot} C_0^\infty(D\Sigma) \hookrightarrow C^\infty(D^*\Sigma)^* \xrightarrow{res^*} C^\infty(D^*M)^* \to C_0^\infty(D^*M)^*$$

where ν is the timelike normal vector to the Cauchy surface, and we claim that this composition actually lands in $C^\infty(DM) \subset C^\infty(D^*M)^*$ and is a classical solution.

Remark 9.4. It's a little strange here that all of the eigenspaces of the Dirac operator are isomorphic (since the left hand side of the above bijection depends on m and the right hand side does not). The Lorentzian signature seems to be doing something interesting here, since this claim is certainly false in the Riemanninan setting.

Theorem 9.5. (1) *Γ has a positive definite inner product given by*

$$\langle u_1, u_2\rangle_\Sigma := \int_\Sigma (\nu \cdot u_1)^\dagger u_2.$$

(2) *$\langle -,-\rangle$ is independent of Σ.*

Remark 9.6. The first part of the above theorem gives the Greens function for the Dirac operator we have.

From the above, we get a Hilbert space $(\overline{\Gamma}, \langle -,-\rangle_\Sigma)$.

9.2. **Quantum Theory.** In general, we want to impose certain "canonical anticommutation relations" on the above Hilbert space. Let B be the unital $*$-algebra generated as:

$$B := \{\chi_0(f), \chi(\alpha) \mid f \in V,\ \alpha \in V^*\}/\{\chi, \chi_0 \text{ are linear},\ \chi(\langle f, -\rangle)^* = \chi_0(f),\ \chi_0(f)\chi(\alpha)+\chi(\alpha)\chi_0(f) = 1{\cdot}\alpha(f)\}$$

Now let

$$\mathsf{CAR}(V, \langle -, -\rangle) := \{\text{eveloping } C^*\text{−algebra of } B\}.$$

So let's do this for the example above by taking $(V, \langle -, -\rangle) := (\overline{\Gamma}, \langle -, -\rangle_\Sigma)$. Now take $f, h \in C_0^\infty(DM)$ and

$$\psi^+(f) := -i\chi_0^\Sigma(res(Sf)) = \psi(f^+)^*$$
$$\psi(h) := \chi^\Sigma(\nu \cdot res(SH))$$

From this we get that

$$\psi(g)\psi^+(f) + \psi^+(f)\psi(h) = -i\int_M hSf dvol.$$

Now we want to actually construct a functor A as per our original definition. So let

$$F(\mathcal{O}) := C^*(\psi(h) \mid \mathrm{supp}(h) \subset \mathcal{O})$$
$$A(\mathcal{O}) := C^*(\psi^+(f)\psi(h) \mid \mathrm{supp}(h),\ \mathrm{supp}(f) \subset \mathcal{O})$$

and $A(\mathcal{O}) \subset F(\mathcal{O})$. It remains to check the axioms on the functor A. But note that F doesn't satisfy the causality axiom.

10. Peter, Free Field Theories III

Today we'll survey some open problems.

Remember we were looking to define these with conformal Dirac operators. We had as our Hilbert space

$$V(Y) := L^2(Y^{d-1}; L^{\frac{d-1}{2}} \otimes S_Y)$$

of spinors tensor $d-1/2$-densities. There is an obvious pairing (since a product will give a $d-1$-density that can be integrate on Y), and $V(Y)$ is a Cl_{d-1}-module. We can package this together to say that this pairing has values in Cl_{d-1}.

This is very close to the setup we need to associate a Heisenberg (or CAR) algebra, expect that rather than the pairing taking values in a (super) commutative algebra, it takes values in Cl_{d-1}.

Question 10.1. *How do we associate a Heisenberg over Cl_{d-1}?*

When $d = 2$, then $Cl_1 = \mathbb{C}$, with grading involution given by complex conjugation. So this gives a $\mathbb{C}$-bilinear form on $V(Y^1)$ via

$$b(v, w) = \langle \alpha(v), w\rangle$$

and since the grading involution is complex antilinear, this turns the hermetian form $\langle -, -\rangle$ into a bilinear form, so we can define $\mathsf{Heis}(V, b)$.

This isn't quite the CAR algebra Ansgar talked about. Instead:

$$\mathsf{CAR} = \mathsf{Heis}(\mathcal{H} \oplus \overline{\mathcal{H}}, b)$$

where

$$\begin{bmatrix} 0 & \langle -, -\rangle \\ \langle -, -\rangle & 0 \end{bmatrix}$$

There are two spin structures on the circle, and we have

$$V(S^1_{per}) \cong C^\infty(S^1; \mathbb{C})$$
$$V(S^1_{aper}) \cong C^\infty(S^1; \text{Mobius} \otimes \mathbb{C})$$

Remark 10.2. In the above we're intentionally confusing Heisenberg algebras and Clifford algebras, since the Heis construction gives Clifford algebra when the bilinear pairing b is symmetric. In general, we assume b is graded-symmetric.

Now,

$$\mathsf{Heis}(V(S^1_{per}), b) \simeq_{Morita} Cl_1, \quad \mathsf{Heis}(V(S^1_{aper}), b) \simeq_{Morita} Cl_0 \cong \mathbb{C}.$$

Now lets build a functor. Recall that the relevant categories are

source $2-\mathsf{CB}^{Spin}$ the 2-category of conformal 1-manifolds, conformal 2-manifolds and conformal isometries.
target $\mathsf{Alg}_{\mathbb{C}}^{\mathbb{Z}/2-\text{graded}}$ of algebras, bimodules and intertwiners (though we'll wait to decide what types of algebras).

Really both of these categories are stacks on manifolds, so we never think about a single 2-manifold, but a family of them, and never a single algebra, but a bundle of them.

Today we'll write down a twist:

$$T(Y^1) := \mathsf{Heis}(V(Y), b), \quad T(\Sigma) := \mathsf{Fock}(\Sigma)$$

and define these free theories in dimension 2.

Question 10.3. *Does this construction extend down to points?*

Recall we have this conformal Dirac operator

$$D^+_\Sigma : C^\infty(\Sigma^2, \mathcal{L}^{1/2} \otimes S^+_\Sigma) \to C^\infty(\Sigma^2, \mathcal{L}^{3/2} \otimes S^+).$$

This operator has a Greens formula

$$\langle D^+\psi, \phi\rangle_\Sigma + \langle \psi, D^+\phi\rangle_\Sigma = b_{\partial\Sigma}(\psi|_\partial, \phi|_\partial)$$

Recall that

$$S^+_\Sigma|_{\partial\Sigma} \cong S_{\partial\Sigma}.$$

So I can pair $\phi, \psi \in C^\infty(\Sigma^2, \mathcal{L}^{1/2} \otimes S^+_\Sigma)$ and get a density to integrate on the boundary, or apply D^+ and pair to get a density I can integrate on Σ.

Denote the restrictions of harmonic spinors, $D^+\psi = 0$ on Σ by $L(\Sigma) \subseteq V(\partial\Sigma)$.

Definition 10.4. $L \subseteq (V, b)$ is Lagrangian if $L \oplus \overline{L} \cong V$, where denotes the grading involution.

Theorem 10.5. *$L(\Sigma) \subseteq V(\partial\Sigma)$ is a b-Lagrangian subspace.*

We used to say that

$$\mathsf{Fock}(\Sigma) = \Lambda^\bullet(\overline{L}(\Sigma)),$$

which is a $T(\partial\Sigma)$-module. Here the elements of L act as creators and $\overline{L}$ as annihilators. The problem is that when we pass to smooth families, there could be jumps in the size of $L(\Sigma)$ (since the kernel of D^+ need not even have the same dimension as we vary the family). We'd really like to have a vector bundle as our family rather than this crazy family. So now we'll give a construction that does this.

First notice that we can form $\mathsf{Fock}(L)$ for any Lagrangian subspace. Furthermore (this is from Pressley and Segal):

$$\mathsf{Fock}^{alg}(L) := \Lambda^{top}(\text{Harmonic spinors on closed components}) \otimes \Lambda^\bullet(\overline{L}) \hookrightarrow Hol(Gr_L(V); Pf^*)$$

where

$$Gr_L(V) := \{L' \subset V \text{ Lagrangian} \mid L \subset V \overset{\text{projection}}{\to} \overline{L} \text{ is Hilbert} - \text{Schmidt}\}$$

This is equivalent (via the Shale-Stinespring-Siegel criterion) to saying that $\mathsf{Fock}(L) \cong \mathsf{Fock}(L')$ as $Cl(V)$-modules. Here we define

$$Pf(L : L(\Sigma)) := \mathsf{Hom}_{Cl(V)}(\mathsf{Fock}(L), \mathsf{Fock}(L(\Sigma)) \hookrightarrow \mathsf{Fock}(L(\Sigma))$$

where the inclusion is induced from letting $Cl(V)$ act on the vacuum vector in $\mathsf{Fock}(L(\Sigma))$. This gives the line of intertwiners between two $Cl(V)$-modules. It will turn out to be related to the Pfaffian of a certain Dirac operator.

Lemma 10.6. *For $L \in Gr_{L(\Sigma)}(V(\partial\Sigma))$ we get a self-adjoint elliptic operator*

$$D^+_L : \{\psi \in C^\infty(\Sigma, \mathcal{L}^{1/2} \otimes S^+_\Sigma) \mid \psi|_\partial \in L\} \to C^\infty(\Sigma^2, \mathcal{L}^{3/2} \otimes S^+_\Sigma)$$

Notice that this is enforcing a certain boundary condition. In particular we will use that $\mathsf{Ker}(D^+_L)$ is finite dimensional.

Lemma 10.7. *$Pf(D^+_L) \cong Pf(L : L(\Sigma))$ canonically.*

Quillen's result is that $Pf(D_L^+)$ varies smoothly in Σ over families (or even holomorphically!), even though the kernel jumps. So now we'll see that even though this kernel isn't well-behaved, the Fock space is. We'll prove this via our new description of the Fock space,

$$\mathsf{Fock}(\Sigma) \cong Hol(Gr_{L(\Sigma)}, Pf(D_L^+)).$$

We're using

$$Pf(D_L^+) \cong \Lambda^{top}(\mathsf{Ker}(D_L^+)) \cong \Lambda^{top}(\mathcal{H}) \otimes \Lambda^{top}(\overline{L} \bigcap L(\Sigma)).$$

(WARNING: The above probably has some typos. We changed our minds about several aspects midway, and I'm not sure that I recorded all of the changes.)

Question 10.8. *Work out all of the issues in passing from Hilbert Fock spaces to nuclear ones. For example, we need the Shale-Stinespring criterion for nuclear spaces. Also we need the right notion of a local model for this Grassmanian in terms of a nuclear complex space.*

Remark 10.9. Notice that diffeomorphisms of 1-manifolds also act (so $T(Y^1)$ gets mapped to an isomorphic Clifford algebra), so we can extend this picture to the "internal category in stacks" formulation.

Now we'd like to write down a weak CFT, or a T-twisted CFT, Q. To do this we actually will need collars! So our bordism category is defined as objects with collars (though apparently is not the version from the most recent Stolz-Teichner survey). We re-interpret our twist functor above as being applied to the core of objects and morphisms.

So let $Y = (Y^c, Y^\pm)$ and define

$$Q(Y) := \varprojlim_{Y^e \to Y^c} {}_{T(Y)}\mathsf{Fock}(Y)_{\mathbb{C}}.$$

We can think of this as a nuclear space, or perhaps the better thing is to remember the pairing and consider this as a rigged Hilbert space.

Lemma 10.10. *There is an isomorphism*

$${}_{T(Y_0)}\mathsf{Fock}({}_{Y_0}\Sigma_{Y_1} \circ_{Y_1} \Sigma'_{Y_2})_{T(Y_2)} \cong {}_{T(Y_0)}\,\mathsf{Fock}(\Sigma) \otimes_{T(Y_1)} \mathsf{Fock}(\Sigma')_{T(Y_2)}$$

and vacuum vectors are respected under this isomorphism $\Omega_{\Sigma \sqcup \Sigma'} \to \Omega_\Sigma \otimes \Omega_{\Sigma'}$.

Question 10.11. *We're very close to being able to say that* Ω_Σ *at a point* L *is the Pfaffian element* $pf(D_L^+) \in Pf(D_L^+)$*, so in particular we get that the vacuum vector is 0 when* D_L^+ *has kernel. Notice that this agrees with the earlier description of the Fock space (without Grassmanians). Working out various duals and grading involutions will make this precise.*

Now we need to define Q on 1-morphisms. So first lets say where it lives:

$$Q({}_{Y_1}\Sigma_{Y_2}) \in \mathsf{Hom}({}_{T(Y_1)}Q(Y_1), {}_{T(Y_1)}\,T(\Sigma) \otimes_{T(Y_0)} Q(Y_0)) = {}_{T(Y_1^c)}\,\mathsf{Fock}(\Sigma^c) \otimes_{T(Y_0^c)} \varprojlim {}_{T(Y_0^c)}\mathsf{Fock}(Y_0).$$

(There is some confusion about this map perhaps going the other way!) By the lemma,

$${}_{T(Y_1^c)}\mathsf{Fock}(\Sigma^c) \otimes_{T(Y_0^c)} \varprojlim {}_{T(Y_0^c)}\mathsf{Fock}(Y_0) \cong \varprojlim {}_{T(Y_1^c)}\mathsf{Fock}(\Sigma)$$

So we have a map

$${}_{T(Y_1)}Q(Y_1) = \varprojlim {}_{T(Y_1^c)}\mathsf{Fock}(Y_1) \to {}_{T(Y_1^c)}\,\mathsf{Fock}(\Sigma^c) \otimes_{T(Y_0^c)} \varprojlim {}_{T(Y_0^c)}\mathsf{Fock}(Y_0)$$

sending

$$v \mapsto v \otimes \Omega_{\Sigma - Y_1}.$$

There are some issues here with the inverse limit in constructing the map.

11. AQFT II, Ansgar

Here's another example of an AQFT, called the free Bose field. Consider a wave equation

$$Pf = 0$$

were P is a generalized Laplacian. Then there exists a unique $E^{\pm} : C_0^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ with the similar properties to the operators S from last time, namely

(1) $PE^{\pm} = E^{\pm}D = i$ where i is the inclusion $C_0^\infty(M) \hookrightarrow C^\infty(M)$.
(2) $\text{supp}(E^{\pm}f) \subset \mathcal{J}^{\pm}(\text{supp} f)$

As before, we call $E := E^+ - E^-$ the propagator of the theory. Furthermore, we have a space

$$\Gamma = \{\text{classical solutions with spacelike compact support}\}$$

and for $\Sigma \subset M$ there is a map

$$\Gamma \to C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$$

where $u \mapsto (u|_\Sigma, \frac{\partial}{\partial \nu} u|_\Sigma)$.

Remark 11.1. Some of us are more used to the propagator being just the formal inverse of P (i.e. just E^+, not a difference). In the case that P is elliptic and M is compact, it seems that we would return to this more familiar setting, as then there is no E^-. The following example is illustrative. Consider the Dirac operator on $\mathbb{R}$. Then we can construct antiderivatives by integrating from 0 to $-\infty$ or from 0 to $+\infty$. The difference is the integral over all of $\mathbb{R}$. If spacetime were compact there wouldn't be these two directions. Conversely, for the 4-manifolds we are considering, $M = \Sigma \times \mathbb{R}$, so we have these limits as $t \to \pm\infty$. There are also some differences in signature: usually we try to solve a heat equation when doing statistical field theories (or "Wick rotated QFTs") whereas now we trying to solve the wave equation.

Notice that Γ has a symplectic form

$$\sigma(u, v) = \int_\Sigma \left(\frac{\partial}{\partial \nu} u \cdot v - u \cdot \frac{\partial}{\partial \nu} v \right) d\Sigma,$$

and σ is independent of Σ by Greens' formula.

Now we can write down the canonical commutation relations of the above symplectic space,

$$CCR(\Gamma, \sigma) := C^*(W(f) \mid W(f) \text{ unitary, } W(f)W(g) = e^{-\frac{i}{2}\sigma(f,g)} W(f,g)).$$

We think of $W(f)$ as $e^{-\phi(f)}$ where $\phi(f) = a^\dagger(f) + a(f)$. So we're writing down the Lie group associated to the the Lie algebra of creation and annihilation operators. The reason we work with the group is that all the operators are bounded, but the Lie algebra contains unbounded operators, which bring in many difficulties. We define our AQFT by

$$A(\mathcal{O}) := C^*(CCR \mid \text{supp}(f) \subset \mathcal{O}).$$

One can verify that these satisfies the axioms.

Definition 11.2. A state ω for A is a natural transformations, $\omega : A \to \mathbb{C}$ where $\mathbb{C}$ is the constant functor.

Equivalently, ω is a state on

$$\omega : \text{colim}_{\mathcal{O}} A(\mathcal{O}) \to \mathbb{C}.$$

Now let's consider some examples of states for the Bose field.

Definition 11.3. A state ω is called regular if all its n-point functions c_n exist:

$$c_n(f_1, \ldots, f_n) := \frac{\partial^n}{\partial t_1 \cdots \partial t_n} |_{t=0} \, \omega(W(tf_1) \ldots W(tf_n))$$

Definition 11.4. A regular state ω is called quasi-free if $c_n = 0$ for n odd and

$$c_{2n}(f_1, \ldots, f_{2n}) = \sum \Pi_{i<j} c_2(f_i, f_j),$$

where the sum is over partitions of pairs.

For example, the 4-point function of a quasi-free regular state is

$$c_4(f_1, f_2, f_3, f_4) = c_2(f_1, f_2)c_2(f_3, f_4) + c_2(f_1, f_3)c_2(f_2, f_4) + c_2(f_1, f_4)c_2(f_2, f_3).$$

Now choose $\mu : \Gamma \times \Gamma \to \mathbb{R}$ some scalar product. Then

$$\omega_\mu(W(f)) := exp(-\frac{1}{2}\mu(f, f)),$$

and extend by linearity to $\omega_\mu : CCR(\Gamma, \sigma) \to \mathbb{C}$. It's clear from the definitions that this function is defined, but it's not clear that we get a state. Recall that ω_μ is a state if $\omega_\mu(A^*A) \geq 0$.

Lemma 11.5. *ω_μ is a state if and only if $\frac{1}{4}(\sigma(f, g))^2 \leq \mu(f, f)\mu(g, g)$.*

Lemma 11.6. *ω_μ is a quasi-free state with $c_2(f, g) = \mu(f, g) + \frac{i}{2}\sigma(f, g)$.*

Applying the GNS construction, we get

$$GNS(\omega_\mu) = (H_\mu, \Omega_\mu, \rho_\mu).$$

Definition 11.7. A one-particle spacefor (Γ, σ, μ) is a Hilbert space over $\mathbb{C}$, $(H_1, \langle -, - \rangle_1)$ and a $\mathbb{R}$-linear map $\iota : \Gamma \to H_1$ such that

(1) $\mathbb{C}\iota\Gamma \subset H_1$ is dense
(2) $\mu(f, g) = \mathrm{Re}(\langle \iota f, \iota g \rangle_1)$
(3) $\sigma(f, g) = 2\mathrm{Im}(\langle \iota f, \iota g \rangle_1)$.

Lemma 11.8. *$(H_1, \langle -, - \rangle_1, \iota)$ exists and is unique.*

Now we get a Fock space

$$H := \bigoplus_{k=0}^{\infty} H_1^{\otimes k}$$

with vacuum $\Omega = (1, 0, 0, \dots)$ and creation and annihilation operators $a^\dagger(x)$, $a(y)$ for $x, y \in H_1$ and

$$\rho_\mu^1 : CCR(\Gamma, \sigma) \to B(H)$$

defines a representation where

$$W(f) \mapsto e^{-a^\dagger(f) + a(f)}$$

and

$$GNS(\omega_\mu) \cong (H, \Omega, \rho_\mu^1).$$

Now note that the vacuum vector we get is not unique (we have a canonical polarization, but not a canonical pairing μ). So from this the notion of particle is also difficult (since a particle is a creator applied to the vacuum). We do get a unique vacuum in spacetimes with a chosen 1-parameter group of isometries, for example Minkowski spacetime. In these situations there is a unique invariant, pure vacuum state.

12. Ansgar, Prelude to DHR Theory

Recall that a globally hyperbolic manifold is a Lorentizian 4-manifold that is oriented and time oriented and has a Cauchy surface (i.e. a codimension 1 surface that hits every inextendable timelike curve exactly once).

Definition 12.1. A globally hyperbolic QFT is a functor

$$A : \{\text{Globally Hyperbolic Manifolds}\} \to C^* - \mathsf{alg}.$$

where the morphisms of manifolds are time orientation preserving and causality preserving isometric embeddings and we take morphisms of C^*-algebras to be inclusions. A causality preserving morphism means that any two points in the domain that can be connected by a timelike curve can also be connected by a timelike curve in the image.

This functor is require to satisfy:

(1) Causality: For two embeddings, $\chi_{1,2} : N_{1,2} \to M$ with the property that $\chi_1(N_1)$ is spacelike seperated from $\chi_2(N_2)$, then the images of $A(\chi_1)$ and $A(\chi_2)$ commute as subalgebras of $A(M)$.
(2) Time-Slice Axiom: For a morphism $\chi : N \to M$ such that $\chi(N) \subset M$ contains a Cauchy surface of M, we require that the induced map $\chi_* : A(N) \to A(M)$ is an isomorphism.

Remark 12.2. The above being a tensor functor is stronger than requiring causality.

We have examples from last time, namely the free scalar QFT and the free Dirac field.

We had a long discussion of the restriction of the above functor to a Cauchy surface, which gives

$$A|_\Sigma : 3 - \mathsf{RB} \to C^* - \mathsf{Alg}.$$

However, we lose information by doing this; the above restriction extends to the category of globally hyperbolic manifolds with a product metric, but not to arbitrary globally hyperbolic manifolds.

Then we discussed some cool representation theory, the Doplicher-Roberts reconstruction theorem (from Inventiones 1989). More on this next time.

13. Ansgar, DHR

I was lazy that day, and didn't take any notes.

14. Harold on Vertex Algebras

Definition 14.1. A vertex algebra consists of data:

(1) a vector space V (of states)
(2) $|0\rangle \in V$ (vacuum)
(3) $T : V \to V$ (translation operator)
(4) a map $Y_z : V \to \mathsf{Hom}(V, V((z))) \subseteq \mathsf{End}(V)[[z^{\pm 1}]]$ (vertex operators). We might call $\mathsf{Hom}(V, V((z)))$ the fields, and this map is a state-field map.

Here $V((z))$ are Laurent series, whereas $\mathsf{End}(V)[[z^{\pm 1}]]$ allows for arbitrary series. For a state $a \in V$, we will use notation:

$$Y_z(a) = Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}.$$

The above data are required to satisfy:

(1) $Y(|0\rangle, z) = Id$, and $\forall a \in V$,
$$\lim_{z \to 0} Y(a, z)|0\rangle = a.$$
(2) $T|0\rangle = 0$, $[T, Y(a, z)] = \partial_z Y(a, z)$
(3) All vertex operators are mutually local:
$$[Y_z(a), Y_w(z)]$$
is "supported on the diagonal," $z = w$, i.e. there is some N such that
$$(z - w)^N [Y_z(a), Y_w(b)] = 0$$
as elements of $\mathsf{End}(V)[[w^{\pm 1}, z^{\pm 1}]]$.

For an example of the final axiom (in the sense of what a zero divisor might look like here):

$$\delta = \sum_{m+n=-1} w^m z^n$$

and we find that

$$(z - w)\delta = 0.$$

Really we should think of this δ as $\delta(z - w)$ since

$$f(z)\delta(z - w) = f(w).$$

In general we should think of these power series as formal distributions. The reason for this is that for a "jet of a function at the origin" i.e. an element of $\mathbb{C}[[z]]$, any linear map to $\mathbb{C}$ can be found by multiplying and taking residues as

$$\mathbb{C}[[z]] \times \mathbb{C}[[z^{\pm 1}]] \to \mathbb{C}.$$

Now notice that for any $a, b, c \in V$,

$$Y_z(a)Y_w(b)c \in V((z))((w)),$$

where one needs to be a bit careful about what $V((z))((w))$ means since for a fixed power of w, we get a Laurent series, but the negative powers in z need not be globally bounded below. We can compare to

$$Y_w(a)Y_z(b)c \in V((w))((z)).$$

We have

$$\begin{array}{ccccc} V((z))((w)) & \leftarrow & V[(z-w)^{-1}] & \rightarrow & V((w))((z)) \\ \delta_- := \frac{1}{z}\sum_{n\geq 0}\left(\frac{w}{z}\right)^n & \leftarrow & \frac{1}{z-w} & \mapsto & \frac{-1}{w}\sum_{n\geq 0}\left(\frac{z}{w}\right)^n =: -\delta_+ \end{array}$$

and we have

$$\delta = \delta_+ - \delta_-$$

so we should think of δ as $\frac{1}{z-w} - \frac{1}{z-w}$, where each of these two were expanded in different ways. We have then that

$$(z-w)^N \partial_w^{N-1}\delta = 0.$$

It is a fact that the commutator in axiom 3 above is actually a sum of derivatives of these formal δ-distributions. More precisely, for an element $v(w,z) \in V[[w^{\pm 1}, z^{\pm 1}]]$ with $(z-w)^N v(w,z) = 0$, then

$$v(w,z) \in \mathsf{Span}_{V[[w^{\pm 1}]]}(\delta, \dots, \partial_w^{N-1}\delta).$$

We don't need to take the full $V[[w^{\pm 1}, z^{\pm 1}]]$ span since v and w act the same way given our assumptions on $v(w,z)$, so actually these spans are the same.

Remark 14.2. Typically we'll also have, $V = \oplus_{k\geq 0} V_k$ with $\mathsf{dim}(V_k) < \infty$. With respect to this we require $\deg(T) = -1$. For $a \in V_k$ then we want $Y(a,z)$ to have conformal dimension k, i.e. for

$$Y(a,z) = \sum a_{(n)} z^{-n-1}$$

we require the degree of $a_{(n)}$ to be $-n+k-1$.

Translating to Segal's thinking this data arises from a chiral CFT, and we can consider the $\mathbb{Z}$ grading on V as coming from the circle action on the Hilbert space. The translation T is the generator of Mobius transformations on the circle that come from the translations on the line, then compactified to the circle. Then T is degree -1 with respect to the rotation action.

14.1. **Holomorphic Vertex Algebra.** Let R be a commutative $\mathbb{C}$ algebra and let $V = R$. Let T be a derivation, $|0\rangle = 1$ and

$$Y(a,z) = e^{zT}a$$

defines a vertex algebra structure. In fact, any vertex algebra where Y_z happens to be a map

$$Y_z : V \to \mathsf{Hom}(V, V[[z]]) \subset \mathsf{End}(V)[[z]]$$

(so-called holomorphic vertex algebras) arise in this way.

14.2. **Heisenberg Vertex Algebra.** Say we have a central extension

$$0 \to \mathbb{C}k \to H \to \mathbb{C}((t)) \to 0$$

We have $\mathbb{C}k \oplus \mathbb{C}[[t]] \hookrightarrow H$. If $b_n := t^n$, then

$$[b_n, b_m] = n\delta^0_{m+n}k.$$

Using this we induce up to H:

$$V = \mathsf{Ind}^H_{\mathbb{C}k\oplus\mathbb{C}[[t]]}\mathbb{C}$$

where on $\mathbb{C}$, $\mathbb{C}[[t]]$ acts by 0 and k acts by 1. As a vector space

$$V \cong \mathbb{C}[b_{-1}, b_{-2}, \dots], \quad \deg(b_{-n}) = n,$$

the bosonic Fock space. We define $|0\rangle = 1$, and T is characterized by

$$T|0\rangle = 0, \quad [T, b_{-n}] = -nb_{-n-1}.$$

Now we have

$$Y(b_{-1}, z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-1}.$$

Remark 14.3. This example is "the free boson." To translate into language from previous talks, $\mathbb{C}((t))$ are holomorphic functions on the punctured disk (or the circle), $\mathbb{C}[[t]]$ are those that extend holomorphically across the disk (so they give the usual Lagrangian, and act by zero on the vacuum). The weird thing is that normally we'd say $\mathbb{C}[[t^{\pm 1}]]$ are holomorphic functions on the punctured disk, but here we don't take all functions. There's probably some algebraic reason for this. But I'm not an algebraist so I can't put my finger on it. The Y_z is what we assign to the pair of pants, or really the Riemann sphere with three punctures.

The action of the Fourier coefficients of $Y(b_{-1}, z)$ on $|0\rangle$ generates V, so Y_z is determined by

Theorem 14.4 (Reconstruction Theorem)**.** *Given V, $|0\rangle$, T and $\{a^i\}_{i\in S} \subset V$ and fields*

$$Y(a^i, z) = \sum a^i_{(n)} z^{-n-1}$$

(i.e. Fourier coefficients of these fields) such that

(1) *The vertex algebra axioms are satisfied "so far" (e.g. locality, action of translation, etc.)*
(2) *V is spanned by $\{a^{i_1}_{(n_1)} \cdots a^{i_k}_{(n_k)}|0\rangle | n_j < 0\}$.*

Then

$$Y(a^{i_1}_{(n_1)} \cdots a^{i_k}_{(n_k)}|0\rangle, z) := \frac{1}{(-n_1-1)! \cdots (-n_k-1)!} : \partial_z^{-n_1-1} Y(a^{i_1}, z) \dots \partial_z^{-n_k-1} Y(a^{i_k}, z) :$$

uniquely extends the given information to a vertex algebra structure.

To understand this, first we'll look at the definition of the normally ordered product in the definition of Y in the theorem. Say

$$f(z) = \dots + f_{-2}z^{-2} + f_{-1}z^{-1} + f_0 + f_1 z + f_2 z^2 + \dots$$

We'll denote $f(z)_-$ to be the sum of all the negative powers, and $f(z)_+$ to be the sum of all the positive powers. Then

$$: \phi(z)\psi(w) := \phi(z)_+\psi(w) + \psi(w)\phi(z)_-.$$

This isn't an associative product in general, so we make the convention

$$: ABC :=: A(: BC :) :$$

Note that given fields $\phi(z)$, $\psi(z)$,

$$\phi(z)\psi(z) = \sum_N \sum_{m+n=N} \phi_m \psi_n z^{-N-2}$$

may not be well-defined, but $: \phi(z)\psi(z) :$ is always again a field.

Returning to the Heisenberg case, there is something left to check, which is that Y is local with itself

$$[Y(b_{-1}, z), Y(b_{-1}, z)] = \partial_w \delta(z-w).$$

Locality is the same as saying that for any $a, b, c \in V$, we have that $Y_z(a)Y_w(b)c$ and $Y_w(b)Y(a)c$ are the respective expansions in $V((z))((w))$ and $V((w))((z))$ of the "same" element of $V[[w,z]][w^{-1}, z^{-1}, (z-w)^{-1}]$.

Theorem 14.5. *In any vertex algebra,*

$$\sum_{n\in\mathbb{Z}} Y_w(a_{(n)}b)c(z-w)^{-n-1}$$

is the expansion of the above element in $V((w))((z-w))$.

If one thinks of $Y_z(a)$ as "multiplication by a," the above theorem is some kind of associativity condition.

Putting this together, we can say that $Y_z(a)$ and $Y_w(b)$ are local if and only if

$$[Y_z(a), Y_w(b)] = \sum_{k=0}^{N-1} \frac{1}{k!} \gamma_k(w) \partial_w^k \delta(z-w)$$

where the $\gamma_k(w)$ are fields. In turn, this is true if and only if

$$Y_z(a)Y_w(b) = \sum_{k=0}^{N-1} \frac{\gamma_k(w)}{(z-w)^k} + : Y_z(a)Y_w(b) :$$

where we understand the first term on the right hand side as its image in $V((z))((w))$ when we expand n negative power of z (since we need an element in $\mathsf{Hom}(V, V((z))((w)))$). Note that the normally ordered product is in $\mathsf{Hom}(V, V[[w,z]][w^{-1}, z^{-1}]$, so make sense without alteration.

Question 14.6. *In some sense, the first term in the above looks like a counterterm, ala Costello. Is this crazy?*

15. Vertex Operator Algebras, Harold

15.1. The Virasoro Vertex Algebra.

We have

$$0 \to \mathbb{C}K \to \mathsf{Vir} \to \mathbb{C}((z))\partial_z \cong \mathsf{Der}(\mathbb{C}((t)))$$

where Vir is the Virasoro algebra. There is an inclusion $\mathbb{C}K \oplus \mathsf{Der}(\mathbb{C}[[t]]) \hookrightarrow \mathsf{Vir}$, and with this notation, we have generators of Vir,

$$-L_n = -t^{n+1}\partial_t$$

with commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^2-n}{12}\delta^0_{m+n}K.$$

The analog of the Fock space here is

$$V_C = Ind^{\mathsf{Vir}}_{\mathbb{C}K \oplus \mathbb{C}[[z]]\partial_z}\mathbb{C}_c \cong \mathbb{C}[L_{-2}, L_{-3}, \dots]$$

where K acts by c on $\mathbb{C}_c$ and $\mathbb{C}[[z]]\partial_z$ acts by 0. We ngrade V_C by declaring $\mathsf{deg}(L_{-n}) = n$. We get a vertex algebra b setting $T = L_{-1}$ and $|0\rangle = |\otimes|$. We set

$$Y(L_{-2}, z) = \sum L_n z^{-n-2},$$

and by the reconstruction theorem from last time, this determines a VA structure once we check that

$$[T(z), T(w)] = \frac{c}{12}\partial^3_w\delta(z-w) + 2T(w)\partial_w\delta(z-w) + (\partial_w T(w))\delta(z-w),$$

and this goes like $(z-w)^4[T(z), T(w)]$, a formal distribution with the desired support.

15.2. Conformal Vertex Algebras.

A conformal VA of charge c is a us a VA V with a choice of $\omega \in V_2$ such that

(1) The Fourier coefficients of
$$Y(\omega, z) = \sum L^V_n z^{-n-2}$$
satisfy Virasoro relations with central charge c.

(2) $L^V_{-1} = T$

(3) $L^V_0|_{V_n} = n \cdot Id$

In the above, all our definitions and construction are tied to some coordinate on the formal disk. We'd really like to get a vertex algebra over $Spec(\mathbb{C}((z)))$, and then a choice of coordinate should return the above data. In turn, this will let us define a vertex algebra for any formal neighborhood on a Riemann surface.

So say $x \in X$, for X a Riemann surface. Let $\overline{\mathcal{O}_x}$ denote the completion of the local ring at x and K_x its ring of fractions. If there is an action of $\mathsf{Aut}(\mathbb{C}[[z]])$ on V, we can form

$$V_x := (V \times \mathsf{Iso}(\mathbb{C}[[z]], \mathcal{O}_z))/\mathsf{Aut}(\mathbb{C}[[z]]),$$

which will remove the choice of coordinate. Note that the automorphisms of $\mathcal{O}_x$ act on V_x, as expected.

Now, the automorphisms will be determined by power series ρ with no constant term:

$$\mathsf{Aut}(\mathbb{C}[[z]]) \cong \{z \mapsto \rho(z)\} \cong \mathbb{C}^\times \ltimes \varprojlim \mathsf{Aut}(\mathbb{C}[[z]]/z^n\mathbb{C}[[z]])$$

and

$$\varprojlim \mathsf{Aut}(\mathbb{C}[[z]]/z^n\mathbb{C}[[z]]) \cong \{z \mapsto z + a_2z^2 + \dots\}.$$

There is an exponential map from $\mathsf{Der}_0(\mathbb{C}[[z]]) \cong z\mathbb{C}[[z]]\partial_z$ onto $\mathsf{Aut}(\mathbb{C}[[z]])$.

For a conformal VA

(1) $z\partial_z$ acts by L^V_0, which is the grading operator. So in particular it has integer eigenvalues.

(2) for $z^n\partial_z$ with $n > 1$, the action of L_{n-1} has negative degree and so acts locally nilpotently. Thus, exponentiation makes sense, and we get an $\mathsf{Aut}(\mathbb{C}[[z]])$ action on V.

We can globalize $\mathsf{Iso}(\mathbb{C}[[z]])$ to a principle $\mathsf{Aut}(\mathbb{C}[[z]])$-bundle and then form the associated bundle

$$(V \times \mathsf{Aut}_\Sigma)/\mathsf{Aut}\mathbb{C}[[z]].$$

15.3. **Primary Fields.** We say that $Y(a, z)$ is a primary field of conformal dimension Δ if A spans an $\mathsf{Aut}(\mathbb{C}[[z]])$ submodule and is degree Δ, i.e.

$$L_0^V A = \Delta A$$

(which implies $L_n^V A = 0$ for $n > 0$ by virtue of A spanning a submodule).

Remark 15.1. In the above, degree and conformal dimension mean the same thing. We don't know where the words "conformal dimension" came from, but they might make the above seem scarier than it really is.

For $\rho(z) \in \mathsf{Aut}(\mathbb{C}[[z]])$, let $R(\rho)$ be the corresponding operator on V.

Theorem 15.2. *Let $Y(A, z)$ be a primary field of conformal dimension Δ. Then*

$$Y(A, z) = R(\rho)Y(A, \rho(z))R(\rho^{-1})(\rho'(z))^\Delta$$

so $Y(A, z)$ is a Δ-form on the formal punctured disk.

The content here is: first we showed we were able to put V on an arbitrary Riemann surface, and now we find that (at least for particular $A \in V$) we can put Y on an arbitrary Riemann surface.

Remark 15.3. In the above theorem, by a Δ-form we mean a section of the Δth *symmetric* power of the canonical bundle.

Remark 15.4. To translate a bit to Segal's thinking, we have a map

$$\mathcal{E}_0 \hookrightarrow \mathsf{Aut}(\mathbb{C}[[z]].$$

where the arrow takes $f \in \mathcal{E}_0$ and takes the germ at zero. This is the sense in which primary fields only depend on the 1-jet. Note, however, that the above map is very, very far from being surjective: power series in $\mathcal{E}_0$ have to converge, and there are growth conditions that guarantee that $f : D^2 \hookrightarrow D^2$

In the above we can replace $\mathsf{Iso}(\mathbb{C}[[z]], \mathcal{O}_x)$ with $\mathsf{Emb}((D^2, 0), (\Sigma, x))$. In the limit as the disks shrink, we get the map described above.

Let E be a functorial CFT and $E(S^1) = H$. Then

$$V_{(D^2,0)} = \lim_{f \in \mathcal{E}_0} E(S^1)$$

and more generally

$$V_{(X,x)} = \lim_{f \in \mathsf{Emb}((D,0),(X,x))} E(S^1),$$

where the disks are embedded in the interior of X, i.e. the disks can't meet the boundary of X (this last condition is so that one can make sense of the half-collars needed to get objects in the bordism category).

There is a map from the limit of this diagram to the colimit, which gives a map $V_{(X,x)} \to E(\partial X)$. This is something like the state-field map when X is an annulus.

The next step is to see what group acts on $V_{(X,x)}$. For example, do we have a $\mathsf{Diff}^+(S^1)$ action? We do on each $E(S^1)$ in the limit, so we need to check some commutative diagrams.

16. Edward Frenkel, Vertex Algebras

This is all from "the" book on vertex algebras. There is also an arxiv note from June 2000 that closely parallels what this talk will be about.

16.1. **Introduction.** From Segal's axioms for a 2D CFT, we have a braided monoidal category $\mathcal{C}$ and a modular functor. This modular functor spits out a vector space for every Riemann surface with some marked points, and this vector space is sometimes called the vector space of conformal blocks. One might ask how these conformal blocks change as one deforms the Riemann surface with marked points. So let $\mathcal{M}_{g,n}$ denote the moduli space of Riemann surfaces with n marked points. Basically on physical grounds, we expect conformal blocks to give an honest vector bundle on $\mathcal{M}_{g,n}$ with a projectively flat connection. With $g = 0$ this is the KZ connection.

The question is how do we construct this bundle with connection?

The first example that we know well is the WZW model. To define $\mathcal{C}$, we start with the data of a simple complex Lie algebra $\mathfrak{g}$ of a complex, connected and simply connected Lie group G. We have the formal loop algebra $\mathfrak{g}((t))$ of formal power series. This gives a Lie algebra with the usual bracket. There is a universal 1-dimensional central extension of this Lie algebra,

$$0 \to \mathbb{C}K \to \hat{\mathfrak{g}} \to \mathfrak{g}((t)) \to 0.$$

We want to consider the level k representations of $\hat{\mathfrak{g}}$. To connect to Segal's picture we want $k \in \mathbb{Z}$. We want to consider the category $\mathcal{C}_k(\mathfrak{g})$ whose objects are "integrable" $\hat{\mathfrak{g}}$-modules of level k. Here "integrable" means that the Lie algebra data integrates, so we get an extension of algebraic groups

$$\mathbb{C}^\times \to \hat{G} \to G((t)).$$

For example, when $G = Sl_n$, we have

$$Sl(2)((t)) = \{M \in M_{n\times n}(\mathbb{C}((t))) : \mathsf{det}(M) = 1\}.$$

We can consider the vacuum module, given $\hat{g}$ of level $k \in \mathbb{C}$. Then

$$V_k(\mathfrak{g}) := Ind_{\mathfrak{g}[[t]]\oplus\mathbb{C}K}\mathbb{C}_k$$

where $\mathbb{C}K$ act on $\mathbb{C}_k$ by k and nonnegative powers of t in $\mathfrak{g}[[t]]$ act on $\mathbb{C}_k$ by 0. When we induce, we start with $v_k \in \mathbb{C}_k$ and apply elements of $\hat{g}$ that aren't in $\mathfrak{g}[[t]]$, and these generate a graded representation, e.g. a part of the degree n part is $J^a t^{-n} v_k$ for $J^a \in \mathfrak{g}$. These $J^a_n := J^a t^n$ are the creation operators for $n < 0$.

We want to understand when this Lie algebra action can be lifted to a group action. This works when k is a non-negative integer. For these, there is some subrepresentation $I_k(\mathfrak{g}) \subset V_k(\mathfrak{g})$ (that can be described explicitly) and the quotient $V/I =: L_k(\hat{\mathfrak{g}})$ is an integral representation. To prove this type of stuff requires some computations with Kac-Moody algebras. Then we have some e_i (annhiliation operators), h_i (energy operators) and f_i (creation operators). The h_i act diagonal, so lifting to a Lie group requires they have integral eigenvalues (which already forces the level to be an integer). The e_i act locally nilpotently automatically. There are some interesting conditions one needs to put on the f_i in order for this to be an integral representation, analogous to the positive energy condition.

Getting back to this vector bundle on $\mathcal{M}_{g,n}$, we may wish to extend this across the Deligne-Mumford stack $\overline{\mathcal{M}_{g,n}}$. In the case of the integral $\hat{\mathfrak{g}}$-modules, we get such an extension.

16.2. **Rational Versus Irrational Vertex Algebras and Their Modules.** The question is: does this picture generalize, and what is the appropriate language in which to generalize it? The first thing to notice is that the $V_k(\mathfrak{g})$ above is a vertex algebra, and $\mathcal{C}_k(\mathfrak{g})$ is precisely the category of all modules over the vertex algebra $L_k(\mathfrak{g})$. So we're going to start this story by replacing $L_k(\mathfrak{g})$ with more general vertex algebras V, and $\mathcal{C}_k(\mathfrak{g})$ will be replaced by the category of V-modules.

From the outset, we should realize that $L_k(\mathfrak{g})$ is a very special vertex algebra.

Now, a module over $V_k(\mathfrak{g})$ is the same as a $\hat{\mathfrak{g}}$-module M of level k with the additional property that for all $v \in M$, there is some $N \in \mathbb{Z}$ such that $t^N\mathfrak{g}[[t]]v = 0$. There are uncountably many of these (they contain the Verma modules) and the conformal blocks associated to these need not be finite dimensional. In contrast, modules over $L_k(\mathfrak{g})$ are automatically $V_k(\mathfrak{g})$-modules, but there are only finitely many $L_k(\mathfrak{g})$-modules, namely the integral $\hat{\mathfrak{g}}$-modules. For these the spaces of conformal blocks are finite dimensional. We call vertex algebras of this sort "rational vertex algebras."

There are interesting examples of "irrational" vertex algebras, but in some sense they don't come from Segal CFTs since they wouldn't give a modular functor (the conformal blocks aren't finite dimensional).

16.3. **After the Break.** Let $V_k(\mathfrak{g})$ be $\hat{\mathfrak{g}}$ modules of level k. Fix $M_1, \ldots, M_N$ $\hat{\mathfrak{g}}$-modules of level k. Then take a Riemann surface X with N marked points x_i and choose coordinates t_i near these points. Then we have the "diagonal" central extension

$$\mathbb{C}K \to \hat{g}_N \to \oplus \mathfrak{g}((t_i)),$$

which is really a quotient of N copies of $\hat{\mathfrak{g}}$. In the above, $\hat{\mathfrak{g}}_N$ acts on $\otimes_{i=1}^N M_i$. We have the algebraic functions on $X - \{x_i\}$ and can consider

$$\mathfrak{g} \otimes \mathbb{C}[X - \{x_i\}] \hookrightarrow \oplus_{i=1}^N \mathfrak{g}((t_i))$$

where we expand at all points. In fact, there is a lift,

$$\mathfrak{g} \otimes \mathbb{C}[X - \{x_i\}] \to \hat{\mathfrak{g}}_N.$$

Let's describe the central extension $\hat{\mathfrak{g}}_N$ more explicitly. Let $A_i \otimes f_i \in \mathfrak{g} \otimes \mathbb{C}((t_i))$. Then

$$[(A_i \otimes f_i(t_i)), (B_i \otimes g_i(t_i))] = [A_i(t_i), B_i(t_i)] - \left(\sum_{i=1}^N \langle A_i, B_i \rangle Res(f_i dg_i) \right) K.$$

How can we play around with this structure? We can take the space of coinvariants,

$$H_{\mathfrak{g},k}(X, x_i, M_i) = \otimes_{i=1}^N M_i / \mathfrak{g}_{out}$$

where here the quotient is by the space of $a \cdot m$ for $a \in \mathfrak{g}_{out}$ and $m \in \otimes M_i$ and

$$\mathfrak{g}_{out} := \mathbb{C}[X - \{x_i\}].$$

We can also take the dual space

$$C_{\mathfrak{g},k}(X, x_i, M_i) := \mathsf{Hom}_{\mathfrak{g}_{out}}(\otimes_{i=1}^N M_i, \mathbb{C})$$

and this is what people usually call the space of conformal blocks. The coinvariants are mathematically a bit easier to handle, though physically the conformal blocks are related to correlation functions, and so perhaps of greater interest to physicists. If we happen to be working in the case where the conformal blocks are finite dimensional, there isn't a huge distinction.

16.4. **Vertex Algebras.** Recall that a vertex algebra $(V, |0\rangle, Y)$ is a vector space V, a vacuum vector $|0\rangle$ and a map

$$Y : V \to \mathsf{End}(V)[[z, z^{-1}]]$$

where the notation we use is

$$v \mapsto Y(v, z) = \sum_{n \in Z} v_{(n)} z^{-n-1}$$

where $v_{(n)} \in \mathsf{End}(V)$. There are some axioms we require, in particular

$$Y(|0\rangle, z) = Id,$$

so $|0\rangle_{(n)} = Id$ for $n = -1$ and 0 otherwise.

Before we had some element v_k on which we applied creators J^a_{-n} an annhilators J^a_{+n}, so there $v_k = |0\rangle$ and

$$Y(J^a_{-1} v_k, z) := \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1} =: J^a(z).$$

There are some other important rules. For example, to make sense out of products we need normal ordering

$$Y(J^a_{-1} J^b_{-1} v_k, z) = \, : J^a(z) J^b(z) :$$

We also note that

$$[\partial_t, J^a_n] = -n J^a_{n-1}$$

so

$$Y(J^a_{-2} v_k, z) = \partial_z J^a(z).$$

Really the important players here are the fields, $J^a(z)$: everything else is obtained by normally ordered products or derivatives of these.

One drawback of the discussion up until now is that everything is defined in terms of coordinates on a Riemann surface. So what happens when we change from coordinates z to coordinates w with $z = f(w)$?

What happens to the fields? More seriously, is there an intrinsic meaning of these fields? It will turn out that

$$J^a(z) \mapsto J^a(w)(f'(w))^{-1},$$

so that $J^a(z)dz$ has intrinsic meaning. To give an idea of why this is a 1-form and not a function is that there is the residue pairing,

$$Res(f(z)J^a(z)dz) = J^a \otimes f(t) \in \mathfrak{g}((t)).$$

The key here will be to look at the group of changes of coordinates, and understand how it acts on the vertex algebra. This will allow us to construct a bundle on arbitrary Riemann surfaces. So let's consider formal changes of coordinates ρ on the formal disk,

$$\mathsf{Aut}(\mathcal{O}) = \rho(z) = a_1 z + a_2 z^2 + \cdots \in \mathbb{C}[[z]],\ a_1 \in \mathbb{C}^\times$$

where $\mathcal{O} = \mathbb{C}[[z]]$. The claim is that there is a natural action of $\mathsf{Aut}(\mathcal{O})$ on $V_k(\mathfrak{g})$ naturally. Now, $\mathsf{Aut}(\mathcal{O})$ acts on $\mathbb{C}((z))$ and so on $\mathfrak{g}((z))$, and so on $\hat{\mathfrak{g}}$ (the second implication comes from understanding the central extension via the residue pairing, which is a coordinate invariant number).

Now $Lie(\mathsf{Aut}(\mathcal{O})) = Der_0(\mathcal{O}) = z\mathbb{C}[[z]]_{\partial_z}$. We can actually exponential the Lie algebra action because $-z\partial_z$ acts as the grading operator, and the higher powers in z acts locally nilpotently, so we can exponentiate.

17. Frenkel, II

Now we'll explain conformal blocks in the general setting. So recall we have

$$Y : V \to \mathsf{End}(V)[[z, z^{-1}]], \quad v \mapsto Y(v, z) = \sum_{n\in\mathbb{Z}} v_{(n)} z^{-z-1}, \quad v_{(n)} \in \mathsf{End}(V),$$

obeying certain axioms. In particular, for any $A \in V$, $Y(v, z)A \in V((z))$, i.e. the arbitrary series in z and z^{-1} is truncated from below.

For example, we had $V_k(\mathfrak{g})$ for $k \in \mathbb{C}$ and $\mathfrak{g}$ some Lie algebra with basis J^a. Then we defined

$$J^a_n := J^a t^n \in \mathfrak{g}((t)) \subset \hat{\mathfrak{g}}.$$

In degree 0, we have the $\mathbb{C}$-span of v_k; in degree 1 we have $J^a_{-1}v_k$; degree 2 we have $J^a_{-2}v_k$ and $J^a_{-1}J^b_{-1}v_k$; etc. So then

$$Y(J^a_{-1}v_k, z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}.$$

Now we'd like to get rid of the dependence on the coordinate z so that we can put this vertex algebra on an arbitrary Riemann surface. For now, we can only put this on the moduli space of formal disks in that surface.

From last time we had $\mathcal{O} = \mathbb{C}[[z]]$ with the z-adic topology. So by continuity, once we know where z goes, we get a unique element ρ of $\mathsf{Aut}(\mathcal{O})$. So let

$$\rho(z) = \rho_0 + \rho_1 z + \rho_2 z^2 + \dots$$

and then since we want an honest automorphism, we find that $\rho_0 = 0$ and $\rho_1 \in \mathbb{C}^\times$. So the $\mathbb{C}$-points of $\mathsf{Aut}(\mathcal{O})$ are:

$$\mathsf{Aut}(\mathcal{O}) = \{\rho(z) | \rho_1 \in \mathbb{C}^\times\}.$$

Remark 17.1. We can actually extend this to a group object in schemes, since

$$\mathsf{Aut}(\mathcal{O})(R) = \{\rho_0 + \rho_1 z + \dots | \rho_0^n = 0 \text{ for some } n,\ \rho_1 \in R^\times\}$$

makes sense for R any $\mathbb{C}$-algebra. These nilpotent directions in the group scheme are important; at the Lie algebra level rather than

$$Lie(\mathsf{Aut}(\mathcal{O})) = \mathsf{Der}_0\mathcal{O} \cong z\mathbb{C}[[z]]\partial_z$$

we find

$$Lie(\mathsf{Aut}(\mathcal{O})) = \mathsf{Der}(\mathcal{O}) \cong \mathbb{C}[[z]]\partial_z.$$

Now we want $\mathsf{Aut}(\mathcal{O})$ and $\mathsf{Der}(\mathcal{O})$ to act on V in a way that is compatible with the vertex algebra structure, so we want a notion of inner automorphisms by which these objects act. By analogy, we have for a Lie algebra $\mathfrak{g}$ that adjoint action of G (or $\mathfrak{g}$) on itself by inner automorphisms. We want to generalize this idea to vertex algebras.

Now notice that

$$\mathsf{Der}(\mathcal{O}) \cong \mathbb{C}[[z]]\partial_z \cong \mathsf{span}\{L_n,\ n \geq -1\}$$

where $L_n = -z^{n+1}\partial_z$, and so $\mathsf{Der}(\mathcal{O}) \subset \mathsf{Der}(\mathbb{C}((z)))$, but

$$\mathsf{Der}(\mathbb{C}((z))) = \mathsf{span}\{L_n,\ n \in \mathbb{Z}\}.$$

So we see that the Lie algebra of automorphisms is something like the loop algebra. But the representation theory of the loop algebra is bad; we need to consider the universal central extension,

$$0 \to \mathbb{C}C \to Vir \to \mathbb{C}((z))\partial_z \to 0.$$

Definition 17.2. A vertex algebra V is called conformal if there exists $T \in V$ such that

$$Y(T,z) = \sum_{n\in\mathbb{Z}} T_{(n)} z^{-n-1}$$

such that $L_n \mapsto T_{(n+1)}$, and $C \mapsto c \cdot Id$ for some $c \in \mathbb{C}$ defines a representation of the Virasoro Lie algebra.

Notice that L_0 defines a grading such that $\mathsf{deg}(v_k) = 0$.

Now,

$$\mathsf{Aut}(\mathcal{O}) \cong \mathbb{C}^\times \rtimes \mathsf{Aut}_+(\mathcal{O})$$

where $\mathbb{C}^\times$ acts by grading, i.e. by $\mathbb{C}L_0$, and

$$\mathsf{Aut}_+(\mathcal{O}) := \{z \mapsto z + \rho_2 z^2 + \dots\} \cong z^2\mathbb{C}[[z]]\partial_z \cong \mathsf{span}\{L_1, L_2, \dots\}.$$

is a pronilpotent group.

Lemma 17.3. *$V_k(\mathfrak{g})$ is conformal if $k \neq -h^\nu$.*

For example, for $\mathfrak{g} = \mathfrak{sl}(n)$, $k \neq -n$. There is the Segal-Sugawara formula for critical level $-h^\nu$,

$$T(z) = \frac{1}{2(k+h^\nu)} \sum_{a=1}^{\dim(\mathfrak{g})} :J_a(z)J^a(z):$$

where J_a and J^a are dual bases with respect to the standard normalization h^ν.

There is a weaker notion of a quasi-conformal vertex algebra, where we require that only $\mathsf{Der}(\mathcal{O})$ act (not the corresponding Lie group) and satisfies some relations with vertex operators similar to the conformal case (see the book joint with Ben-Zvi for more on this).

In a conformal vertex algebra, we use the axioms to write

$$[L_n, Y(A,z)] = \sum \dots Y(L_m A, z).$$

These identities allow us to interpret the operation Y in a coordinate independent way.

So consider $\mathcal{O}_x$ the completed local ring for a point x in a Riemann surface. So far, we need to choose a coordinate $t_x \in \mathcal{O}_x$ to put the vertex algebra V on the surface via the isomorphism $\mathcal{O}_x \cong \mathbb{C}[[t_x]]$. To get rid of this, we need to use the automorphisms.

So let Aut_x be the set of all formal coordinates at $x \in X$. Then $\mathsf{Aut}(\mathcal{O})$ acts on it from the right (so Aut_x is an $\mathsf{Aut}(\mathcal{O})$ torsor, since for $\rho \in \mathsf{Aut}(\mathcal{O})$, we have

$$\mathcal{O}_x \cong \mathbb{C}[[t_x]] \cong \mathbb{C}[[\rho(t_x)]].$$

Define

$$\mathcal{V}_x := \mathsf{Aut}_x \times_{\mathsf{Aut}\mathcal{O}} V,$$

and now as we vary x we actually get a vector bundle on the Riemann surfaces: there is a principle $\mathsf{Aut}(\mathcal{O})$-bundle,

$$\mathsf{Aut}_X := \{x \in X, t_x \in \mathsf{Aut}_x\}.$$

Then the vector bundle is

$$\mathcal{V} := \mathsf{Aut}_X \times_{\mathsf{Aut}\mathcal{O}} V.$$

This is a fairly standard thing to do, only we have infinite dimensional groups and fibers.

Remark 17.4. On this bundle we still have actions of $\mathcal{O}_x$, which is part of the gauge group of this bundle.

Remark 17.5. If we think about the action of $\underline{\mathsf{Aut}}(\mathcal{O})$ (i.e. the automorphism group with nilpotent directions) we see there is an action of $-\partial_z$ on this space. This is the data of a flat connection. Another way to say this is that at a point in the Riemann surface we don't have a canonical vector field ∂_z, but on the total space of points with choice of coordinate, there is a vector field.

This connection is a bit reminiscent of the one Kevin Costello uses in defining some curved L_∞ algebra coming from affine coordinate charts on a complex manifold.

So in total, we get $\mathcal{V}$ a vector bundle on X with a flat connection (the flatness is automatic since we're on a 1-dimensional complex surface). Let $D_x = Spec\mathcal{O}_x \subset X$, for X the Riemann surface (or algebraic curve). Now given $\mathcal{V}|_{D_x}$, the $\mathcal{V}_x$ fiber at x and

$$Y : V \to \mathsf{End}V[[z^{\pm 1}]]$$

we get a canonical section $\mathcal{Y}_x$ of $\mathcal{V}^*|_{D_x}$ with values in $\mathsf{End}(\mathcal{V}_x$. Moreover, $\mathcal{Y}_x$ is horizontal. Notice that we've lost the grading since we no longer have an action of L_0. However, we do have a filtration

$$0 \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \dots$$

where each piece is finite dimensional. We define $\mathcal{V}^* = \lim \leftarrow \mathcal{V}_i^*$.

Remark 17.6. This feels like gluing together lots of free field theories on a Riemann surface into a non-free theory.

A section of $\mathcal{V}^*$ with values in $\mathsf{End}(\mathcal{V}_x)$ is equivalent to a rule which assigns

(1) a function on $D_x^\times$ to any section of $\mathcal{V}|_{D_x}$ in a $\mathcal{O}_x$-linear fashion;
(2) a vector in $\mathcal{V}_x$ in a $\mathbb{C}$-linear fashion;
(3) and a linear functional on $\mathcal{V}_x$ in a $\mathbb{C}$-linear fashion.

Pick a formal coordinate t_x at x, and identify $\mathcal{V}_x \cong \mathcal{V}$. Then $\mathcal{V}|_{D_x} \cong$ the trivial bundle on D_x with fiber $\mathcal{V}$. We get

(1) a section of $\mathcal{V}$ via $f(t_x)A$ for $A \in V$,
(2) a vector v_λ , $v \in V$
(3) A linear functional on V_x, $\phi : V \to \mathbb{C}$

Then

$$(f(t_x)A, v, \phi) \mapsto \phi(Y(A, t_x) \cdot v) \in \mathbb{C}((t_x))$$

so is a function on $D_x^\times$.

Theorem 17.7. *This section is independent of the choice of* t_x.

So far, we've moved from working on a formal disk with coordinate to a formal disk without coordinate in some Riemann surface. Eventually, we want to globalize to the entire Riemann surface.

Let's look at an example first, $V_k(\mathfrak{g})$. Let's consider $J^a_{-1}v_k \cong \mathfrak{g}_{-1}$, which is preserved by $\mathsf{Aut}(\mathcal{O})$ (which is a computation). So we have

$$\mathsf{Aut}_X \times_{\mathsf{Aut}\mathcal{O}} \mathfrak{g}_{-1} \subset \mathcal{V}_k(\mathfrak{g}),$$

a one-dimensional representation of $\mathsf{Aut}(\mathcal{O})$ on which $\mathbb{C}^\times$ act by $u \mapsto u^N$ (via L_0). All other L_k for $k \geq 0$ act by zero. This bundle is exactly $\Omega^{\otimes -N}$. This gives a surjective map from sections of $\mathcal{V}_k(\mathfrak{g})^*|_{D_x^\times}$ to $\mathfrak{g}^* \otimes \Omega|_{D_x^\times}$ for all $J^a \in \mathfrak{g}$. In a specific coordinate chart we get a 1-form

$$J^a(z)dz = \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1} dz.$$

and if ρ is a change of coordinates we find the action is

$$\rho(J^a(\rho(w))\rho'(w))\rho^{-1} = J^a(w)$$

where ρ without the variable w denotes its action on $V_k(\mathfrak{g})$. So we see that the above 1-form in z-coordinates transforms to a 1-form in the w-coordinates.

Now to describe the connection, we have

$$Y(L_{-1}A, z) = \partial_z Y(A, z),$$

and with $\nabla = d + L_{-1} \otimes dz$, and $\nabla^* = \partial_z - L_{-1}$, we have

$$(\partial_z - L_{-1})Y(A, z) = 0.$$

Let $\mathcal{Y}_x$ be a section of $\mathcal{V}_x^*|_{D_x^\times}$ with values in $\mathsf{End}(\mathcal{V}_x)$. So if $\phi : \mathcal{V}_x \to \mathbb{C}$ and $A \in \mathcal{V}_x$, then $\phi \circ \mathcal{Y}_x$ is a section of $\mathcal{V}_x^*|_{D_x}$.

Definition 17.8. ϕ is called a conformal block if $\phi \circ \mathcal{Y}_x^{(A)}$ extends to a section of $\mathcal{V}_x^*$ on $X - x$ for any $A \in \mathcal{V}_x$.

We have the map $\mathcal{Y}_x \to$ 1-forms "$J^a(z)dz$" with values in $\mathsf{End}(\mathcal{V}_k(\mathfrak{g})_x$. Let $\phi : \mathcal{V}_k(x) \to \mathbb{C}$ be a conformal block. Then for all $A \in \mathcal{V}_k(\mathfrak{g})_x$, $\phi(J^a(x) \cdot A)dz$ extends to $X - x$:

$$\phi((J^a \otimes f) \cdot A) = Res_x f(z)\phi(J^a(z) \cdot A)dz = 0$$

for f a function on $X - x$, and $\phi(J^a(z) \cdot A)dz$ a one-form on $X - x$. Equivalently $\phi(\mathfrak{g}_{out} \cdot A) = 0$, where

$$\mathfrak{g}_{out} \cong \mathfrak{g} \otimes \mathbb{C}[X - x].$$

Department of Mathematics, University of California, Berkeley, 94720
E-mail address: devans@math.berkeley.edu