A SYZYGETIC APPROACH TO THE SMOOTHABILITY OF 0-SCHEMES OF REGULARITY TWO

DANIEL ERMAN AND MAURICIO VELASCO

Abstract. We consider the question of which zero dimensional schemes deform to a collection of distinct points; equivalently, we ask which Artinian k-algebras deform to a product of fields. We introduce a syzygetic invariant which sheds light on this question for 0-schemes of regularity two. This invariant imposes obstructions for smoothability in general, and it completely answers the question of smoothability for certain 0-schemes of low degree. The tools of this paper also lead to other results about Hilbert schemes of points, including a characterization of nonsmoothable 0-schemes of minimal degree in every embedding dimension $d \geq 4$.

1. Introduction

A fundamental question in the study of 0-dimensional schemes is to determine which 0-schemes deform to a collection of distinct points[1], that is, which 0-schemes are smoothable (c.f. [Iar72], [IE78],[Fog68], [Maz80], [Sha90], [Eva04], [CEVV]). For embedding dimension greater than two, very little is known about how to answer this question. In this paper, we introduce a syzygetic invariant which yields new and remarkably sharp information about this question. Our invariant imposes necessary conditions for smoothability of 0-schemes of regularity two, and it completely determines the question of smoothability in low degree.

Previous work on smoothability focuses on tangent space dimension. Since the dimension of the first order deformation space of a 0-scheme $\Gamma \subseteq \mathbb{A}^d$ is upper semicontinuous, having a "small tangent space" poses an obstruction to smoothability. This notion of a "small tangent space" obstruction is introduced and exploited in [IE78], where the graded structure of the tangent space is also used to show that a generic homogeneous ideal with Hilbert function $(1, 4, 3)$ is nonsmoothable. Shafarevich greatly expanded on these results by a similar "small tangent space" obstruction in [Sha90].

Despite the significant results of [IE78] and [Sha90], tangent space dimension is a rather coarse invariant in the study of smoothability. There exist many possible causes for an increase in the number of first order deformations, and these are not necessarily related to smoothability. For instance, if a 0-scheme belongs to the intersection of two irreducible components of the Hilbert scheme, then this 0-scheme will have a large deformation space, but it may not be smoothable (c.f. Example 1.7 (2)).

The invariant introduced below imposes obstructions to smoothability for homogeneous 0-schemes of regularity two and, in some cases, provides even richer information. For instance, our completely answers the question of smoothability for certain0-schemes of low degree (c.f. Theorem 1.4).

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[1] An equivalent question is to determine which Artinian k-algebras deform to k^n.

1.1. **The κ-vector and obstructions to smoothability.** The invariant introduced in this paper is the κ-vector of a homogeneous ideal. We work over an algebraically closed field k with $\mathrm{char}(k) \neq 2, 3$. We say that a 0-scheme Γ has regularity two if $H^0(\Gamma, \mathcal{O}_\Gamma)$ is a local ring whose maximal ideal $\mathfrak{m}$ satisfies $\mathfrak{m}^3 = 0$. Since 0-schemes of regularity two are one of the simplest classes containing infinitely many distinct isomorphism types, it is natural to focus on these families.

Every 0-scheme of regularity two and embedding dimension d admits an embedding $\Gamma \subseteq \mathbb{A}^d$ such that Γ is represented by a homogeneous ideal $I \subseteq S := k[x_1, \ldots, x_d]$. Conversely, every embedding of Γ into $\mathbb{A}^d$ is, up to translation, defined by a homogeneous ideal I. Note that the deformation theory of an embedded 0-scheme is smooth over the abstract deformation theory of the 0-scheme [Art76, p. 4]. Hence we may fix such an embedding of Γ without affecting its deformation theoretic properties. Let $e = \deg(\Gamma) - d - 1$ and let $I_2^\perp \in Gr(e, S_2^*)$ be the degree two part of the (Macaulay) inverse system of the ideal I. Choose a basis $q_1, \ldots, q_e$ of $I_2^\perp$ and represent these elements by $d \times d$-symmetric matrices $A_1, \ldots, A_e$. We define $\kappa_j(I)$ to be the rank of the following linear map induced by $\mathbf{A} = (A_1, \ldots, A_e) \in S_2^* \otimes I_2^\perp$ (see §4 for a more detailed discussion and for an equivalent definition via syzygies):

$$S_1 \otimes \bigwedge^j \left(I_2^\perp\right) \xrightarrow{\wedge \mathbf{A}} S_1^* \otimes \bigwedge^{j+1} \left(I_2^\perp\right).$$

More concretely, when $e = 3$, the numbers $\kappa_0(I), \kappa_1(I), \kappa_2(I)$ are the ranks of the matrices appearing in the following sequence:

$$k^d \xrightarrow{\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}} k^{3d} \xrightarrow{\begin{pmatrix} 0 & A_3 & -A_2 \\ -A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{pmatrix}} k^{3d} \xrightarrow{\begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix}} k^d.$$

Variations of κ_1 have appeared in several geometric settings. Namely, when $e = 3$ the invariant κ_1 is determined by the Strassen equation, and it was previously studied in connection with secant varieties [Ott07], [AB08, p. 14], vector bundles [Ott07], and polynomial versions of Waring's problem [CC03, p. 513].

Definition 1.1. *The κ-vector of I is the sequence $\kappa(I) = (\kappa_0(I), \ldots, \kappa_{e-1}(I))$.*

The κ-vector of I is independent of the choice of basis of $I_2^\perp$ and is invariant under the $GL(d)$-action on S_2^*. Further, the action of $GL(d)$ is transitive on the homogeneous ideals I which define some embedding of $\Gamma \subseteq \mathbb{A}^d$. Hence each $\kappa_j(I)$ is in fact an invariant of Γ itself, and we thus refer to $\kappa_j(\Gamma)$ and $\kappa_j(I)$ interchangeably. The lower semincontinuity of κ_j induces obstructions to the existence of deformations among algebras.

In Proposition 4.3, we compute the values of the κ-vector for generic ideals and generic smoothable ideals with a given embedding dimension and degree. This computation motivates the introduction of the following two conditions:

$$(*) \qquad \kappa_j(\Gamma) \leq (d + e)\binom{e-1}{j}$$

for $j = 1, \ldots, e - 1$ and

$$(**) \qquad \kappa_1(\Gamma) \leq (e - 1)d + \binom{e}{2}.$$

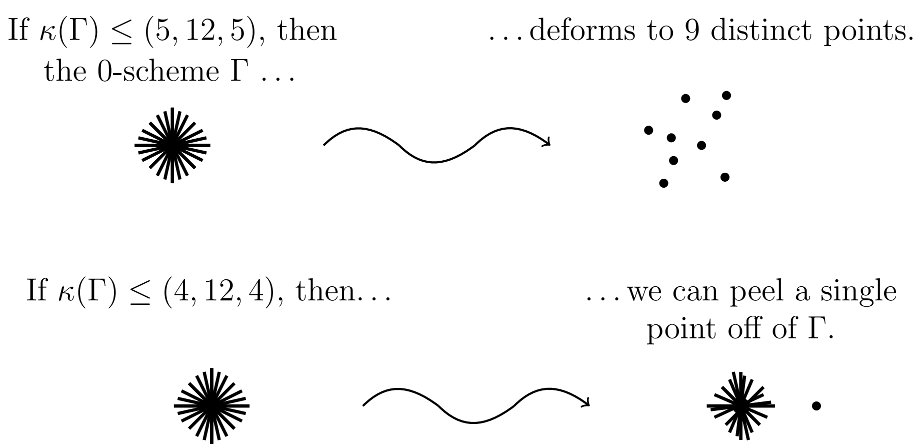

Figure 1. In Example 1.7(3) we consider a 0-scheme Γ of regularity two, embedding dimension 5, and degree 9. The example illustrates how the κ-vector contains rich information about the deformations of Γ.

Theorem 1.2. *Let $\Gamma \subseteq \mathbb{A}^d$ be a 0-scheme of regularity two, embedding dimension d, and degree $1 + d + e$. Then (*) and (**) are necessary conditions for the smoothability of Γ.*

The above obstructions are nontrivial when $e \geq 3$ (c.f Example 1.7 (1)).

1.2. **Minimal examples and sufficient conditions for smoothability.** We next consider minimal examples of nonsmoothable 0-schemes. The results of [CEVV] imply that every minimal degree nonsmoothable 0-scheme embeds into $\mathbb{A}^4$ and is, up to translation, defined by a homogeneous ideal $I \subseteq k[x_1, \ldots, x_4]$ with Hilbert function $(1, 4, 3)$. We extend this result by characterizing, for any $d \geq 4$, the minimal degree nonsmoothable 0-schemes which cannot be embedded in $\mathbb{A}^{d-1}$. The proof of the following theorem relies heavily on results from [CEVV].

Theorem 1.3. *For $d \geq 4$, let $\Gamma \subseteq \mathbb{A}^d$ be a minimal degree subscheme of $\mathbb{A}^d$ which is not smoothable and which cannot be embedded in $\mathbb{A}^{d-1}$. Then, up to translation, Γ is defined by a homogeneous ideal $I \subseteq S$ with Hilbert function $(1, d, 3)$. Moreover, any sufficiently generic homogeneous ideal with Hilbert function $(1, d, 3)$ defines a nonsmoothable 0-scheme.*

When considering a family whose generic element is nonsmoothable, it is natural to seek sufficient conditions for the smoothability of a specific member of the family. To the authors' knowledge, the only previously known set of nontrivial equations for computing the smoothability of a 0-scheme is given in [CEVV, Thm. 1.3], where the case of Hilbert function $(1, 4, 3)$ is considered. By focusing on the κ-vector, we extend [CEVV, Thm. 1.3] to new families of 0-schemes.

Theorem 1.4. *Assume that $char(k) = 0$. As in Theorem 1.3, let $\Gamma \subseteq \mathbb{A}^d$ be a 0-scheme of regularity two, embedding dimension d and degree $d + 4$. In the following cases, (*) and (**) are sufficient conditions for the smoothability of Γ:*

(1) When $d \leq 8$.
(2) When $d \leq 11$ and $I_2^{\perp}$ contains a nonsingular quadric.

If $d \geq 12$, then conditions (*) and (**) are not sufficient to guarantee the smoothability of Γ.

Remark 1.5. The inequalities (*) and (**) may be viewed as inducing determinantal equations for the intersection of components of the Hilbert scheme of points. For instance, the set of homogeneous ideals $I \subseteq S$ with Hilbert function $(1, d, 3)$ is parametrized by the Grassmanian $\mathrm{Gr}(3, S_2^*)$. Theorem 1.4 shows that, when $d \leq 8$, the determinantal equations induced by κ_1 precisely cut out the intersection of $\mathrm{Gr}(3, S_2^*)$ with the smoothable component of the Hilbert scheme of $d + 4$ points in $\mathbb{A}^d$.

When $d = 4$, there is an interesting comparison with the tangent space obstruction. In this case, κ_1 determines a single equation P on $\mathrm{Gr}(3, S_2^*)$. As illustrated in [CEVV], the tangent space obstruction of [IE78] also induces a determinantal equation Q on $\mathrm{Gr}(3, S_2^*)$, and [CEVV, Lemma 5.17] shows that P is irreducible and that $Q = P^8$.

1.3. **Further results on Hilbert schemes of points.** The tools introduced in this paper lead to other new results about Hilbert schemes of points. The results below are phrased for the Hilbert scheme of points in $\mathbb{A}^d$, but the obvious analogues also hold for the Hilbert scheme of points in $\mathbb{P}^d$.

Proposition 1.6. *Let R_n^d be the smoothable component of the Hilbert scheme of n points in $\mathbb{A}^d$ with $d \geq 11$ and $n \geq 15$. There exists a closed subset $Z \subseteq R_n^d$ of codimension 1 such that every point of Z is a singular point of the Hilbert scheme.*

Example 1.7. *Assume that char$(k) = 0$.*

*(1) (***Nonsmoothable Families:***) If $e \geq 3$ and $d > \binom{e}{2}$, then a generic 0-scheme of regularity two, embedding dimension d, and degree $1 + d + e$, is not smoothable.*

*(2) (***Intersections away from the smoothable component***): The Hilbert scheme of 11 points in $\mathbb{A}^7$ contains two components which intersect away from the smoothable component.*

*(3) (***9 Points in $\mathbb{A}^5$***): The κ-vector yields detailed information abou the deformations of 0-schemes of degree 9 in $\mathbb{A}^5$. We say that $J \subseteq k[x_1, \ldots, x_5]$ is a $(1, 4, 3)^{+1}$-ideal if it is the intersection of a homogeneous ideal with Hilbert function $(1, 4, 3)$ and the ideal of a reduced point. Let $I \subseteq k[x_1, \ldots, x_5]$ be a homogeneous ideal with Hilbert fucntion $(1, 5, 3)$ defining a 0-scheme Γ. Then we have*

Γ *deforms into a ...*	*if and only if ...*
union of 9 distinct points	$\kappa(I) \leq (5, 12, 5)$
$(1, 4, 3)^{+1}$-*ideal*	$\kappa(I) \leq (4, 12, 4)$
smoothable $(1, 4, 3)^{+1}$-*ideal*	$\kappa(I) \leq (4, 10, 4)$

Remark 1.8. Both [IE78] and [Sha90] consider tangent space obstructions to the smoothability of 0-schemes of regularity two. The strongest result along these lines is [Sha90, Thm. 2] which implies that, if Γ is a generic 0-scheme satisfying the conditions of Theorem 1.2, then Γ is nonsmoothable whenever $2 < e \leq \frac{(d-1)(d-2)}{6} + 2$. Note that Shafarevich's result is strictly stronger than our Example 1.7 (1).

1.4. **Outline of paper.** The material in this paper is organized as follows. Notation and background on Hilbert schemes, inverse systems, and other topics is included in §2. In §3, we present a dominant rational map to the smoothable regularity two ideals parametrized $\mathrm{Gr}(e, S_2^*)$. In §4, we elaborate on the definition of the κ-vector of an ideal, and we compute values of the κ-vector for generic and generic smoothable ideals of regularity two. We also introduce a module whose graded Betti numbers encode the κ-vector of an ideal. In §5 we introduce κ-cycles, which are $GL(d)$-equivariant subsets of $\mathrm{Gr}(e, S_2^*)$ defined in terms of the κ-vector, and which play a role in the proof of Theorem 1.4. In §6, we combine the results of the earlier sections to prove Theorems 1.2, 1.3, and 1.4. Finally, in §7, we discuss further connections between deformations of 0-schemes of regularity and the κ-vector, and we present the results listed in Example 1.7.

2. Background and Notation

2.1. **Hilbert schemes.** We use the notation H_n^d for the Hilbert scheme of n points in $\mathbb{A}^d$, and we let R_n^d stand for the smoothable component of H_n^d. We now discuss a coordinate system on H_n^d. The reader should refer to [MS05, Ch. 18] for details. Given a monomial ideal M of colength n with standard monomials λ, let $U_\lambda \subseteq H_n^d$ be the set of ideals I such that the monomials in λ are a basis for S/I. Note that the sets U_λ form an open cover of H_n^d. An ideal $I \in U_\lambda$ has generators of the form $m - \sum_{m' \in \lambda} c_{m'}^m m'$. The $c_{m'}^m$ are local coordinates for U_λ which define a closed immersion into affine space.

2.2. **Inverse systems.** Let V be the vector space $V = \langle x_1, \ldots, x_d \rangle$. The symmetric algebra $\mathrm{Sym}^\bullet(V)$ is isomorphic to the polynomial ring $S = k[x_1, \ldots, x_d]$ with the usual grading. We define S^* to be the divided power algebra $\mathrm{Div}^\bullet(V^*)$. The ring S^* is a graded algebra and there is a perfect pairing $S_j \times S_j^* \to S_0^* = k$.

Via this perfect pairing, it is equivalent to give a subspace $I_i \subseteq S_i$ or its orthogonal subspace $I_i^\perp \subseteq S_i^*$. If I is a homogeneous ideal in S then we set $I^\perp = \oplus I_j^\perp$. The space $I^\perp$ is called the (Macaulay) inverse system of the ideal I. Let $y_1, \ldots, y_d$ a basis of V^* which is dual to $x_1, \ldots, x_d$. In characteristic 0, the ring S^* is isomorphic to the polynomial ring $T := k[y_1, \ldots, y_d]$. Further, if $\mathrm{char}\, k = p$, then $T_i \cong S_i^*$ for all $i < p$. Since we assume $\mathrm{char}\, k \neq 2, 3$ and focus on ideals of regularity two, we will abuse notation and identify S^* and T throughout. The reader may refer to [EI95, §2] and [Eis95, §A2.4] for further details.

2.3. **Homogeneous ideals of regularity two.** We often consider ideals $I \subseteq S$ which are homogeneous and which have Hilbert function $(1, d, e)$.

Definition 2.1. *If $I \subseteq S$ is homogeneous and has Hilbert function $(1, d, e)$, then we refer to I as a $(1, d, e)$-ideal. Note that every zero-dimensional homogeneous ideal of regularity two is a $(1, d, e)$-ideal where d is the embedding dimension of I, and where $e = deg(I) - d - 1$.*

The $(1, d, e)$-ideals are parametrized by $\mathrm{Gr}(e, S_2^*)$ in the following way. Given a $(1, d, e)$-ideal, observe that $I_2^\perp \in \mathrm{Gr}(e, S_2^*)$. Conversely, given $V \in \mathrm{Gr}(e, S_2^*)$, the ideal $\langle V^\perp \rangle + \mathfrak{m}^3$ defines a unique $(1, d, e)$-ideal. By abuse of notation, we will generally consider $\mathrm{Gr}(e, S_2^*)$ to be the subscheme of H_{1+d+e}^d which parametrizes $(1, d, e)$-ideals of S. If I is a $(1, d, e)$-ideal, we often write $I \in \mathrm{Gr}(e, S_2^*)$ in place of $I_2^\perp \in \mathrm{Gr}(e, S_2^*)$.

2.4. **Betti tables and Boij-Söderberg theory.** Given a graded S-module M, let $\mathbb{F}$ be the graded minimal free resolution of M. The graded Betti numbers of M are the integers $b_{i,j}$ defined by $\mathbb{F}_i = \bigoplus_j S(-j)^{b_{i,j}}$. The Betti diagram of M, denoted $\beta(M)$, is the matrix

$$\beta(M) := \begin{pmatrix} b_{0,0} & b_{1,1} & \ldots & b_{p,p} \\ b_{0,1} & b_{1,2} & \ldots & b_{p,p+1} \\ \vdots & \vdots & & \vdots \\ b_{0,r} & b_{1,1+r} & \ldots & b_{p,p+r} \end{pmatrix}$$

Boij-Söderberg theory provides an algorithm for expressing the Betti diagram of a module as a positive rational combination of simple building blocks called pure diagrams. See the introduction of [ES07] for an overview.

3. Smoothable ideals with Hilbert function $(1, d, e)$.

In this section we describe the locus of smoothable $(1, d, e)$-ideals in two steps. First, we show that this locus is irreducible, and that it is dominated by a rational map π from the smoothable component of the Hilbert scheme of points (Proposition 3.1). Second, we give a more concrete description of the image of π (Proposition 3.2). This description will be used in §4 to compute the κ-vectors of smoothable $(1, d, e)$-ideals.

Proposition 3.1. *Let $\pi : R^d_{1+d+e} \dashrightarrow Gr(e, S_2^*)$ be the rational map given by $\pi(J) = \mathrm{in}_{(1,\dots,1)}(J)$, the initial ideal of J with respect to the weight $(1, \dots, 1)$. The locus of smoothable $(1, d, e)$-ideals is the image of π.*

Proposition 3.2. *Let $J \subseteq S$ be a generic smoothable ideal of colength $1 + d + e$, where $e \leq \binom{d+1}{2}$. Let $I = \mathrm{in}_{(1,\dots,1)}(J)$. Then:*

(1) I has Hilbert function $(1, d, e)$. Thus it is completely determined by the e-dimensional vector space $I_2^\perp$.

(2) Up to the action of $GL(d)$, there exist $a_i^{(j)} \in k$ with $1 \leq i \leq d$ and $1 \leq j \leq e$ such that

$$I_2^\perp = \langle q_1, \dots, q_e \rangle$$

with $q_j = \sum_{i=1}^d a_i^{(j)} y_i^2 - \left(\sum_{i=1}^d a_i^{(j)} y_i\right)^2$.

Proof of Proposition 3.1. Let U_e be the union of the monomial patches U_λ such that $U_\lambda \cap \mathrm{Gr}(e, S_2^*) \neq \emptyset$. We will show that the function $J \mapsto \mathrm{in}_{(1,\dots,1)}(J)$ is regular in U_e. If $J \in U_\lambda$ and λ has Hilbert function $(1, d, e)$ then for every $m \notin \lambda$ we have $m - \sum_{m' \in \lambda} c^m_{m'} m' \in J$. Therefore $\mathrm{in}_{(1,\dots,1)}(J)$ contains an ideal I generated by all cubic monomials and $\binom{d+1}{2} - e$ linearly independent quadrics. Thus $I = \mathrm{in}_{(1,\dots,1)}(J)$ since any such I has colength $1 + d + e$. It follows that $\mathrm{in}_{(1,\dots,1)}(J) \in Gr(e, S_2^*)$. On U_e, π is locally a projection away from those $c^{m'}_m$ with $\deg(m') \neq \deg(m)$ and thus regular.

For the second statement, observe that $R^d_{1+d+e} \cap \mathrm{Gr}(e, S_2^*)$ belongs to U_e and that π is the identity on $R^d_{1+d+e} \cap \mathrm{Gr}(e, S_2^*)$. Hence we have

$$R^d_{1+d+e} \cap \mathrm{Gr}(e, S_2^*) \subseteq \pi(U_e \cap R^d_{1+d+e}) \subseteq R^d_{1+d+e} \cap \mathrm{Gr}(e, S_2^*),$$

and we conclude that the image of π equals the locus of smoothable $(1, d, e)$-ideals. $\square$

Let J be a generic ideal of $1 + d + e$ reduced points in $\mathbb{A}^d$, where $e \leq \binom{d+1}{2}$. Acting with translations and with $GL(d)$ we may assume that $V(J)$ contains the origin p_0 and the d canonical basis vectors $p_1, \dots, p_d$ in $\mathbb{A}^d$. Moreover $V(J)$ contains e additional points $p_{d+j} = (a_1^{(j)}, \dots, a_d^{(j)})$, $1 \leq j \leq e$. Let $\widetilde{J}$ be the homogenization of J in $S[x_0]$. Note that $\widetilde{J}$ is the homogeneous ideal defining $\cup_{i=0}^{d+e} [1 : p_i] \subseteq \mathbb{P}^d$.

Lemma 3.3. *With notation as above, we have*

(1) For $1 \le i \le d$, $1 \le j \le e$, and any $s \in \mathbb{N}$, the inverse system $\widetilde{J}^\perp$ contains

$$y_0^s, (y_0 + y_i)^s, (y_0 + a_1^{(j)} y_1 + \dots a_d^{(j)} y_d)^s.$$

(2) $\widetilde{J}_2^\perp \cap k[y_1, \dots, y_n]$ contains $q_1, \dots, q_e$ where

$$q_j = \sum_{i=1}^d a_i^{(j)} y_i^2 - \left(\sum_{i=1}^d a_i^{(j)} y_i \right)^2 .$$

Moreover the q_j are linearly independent.

Proof. For a point $r = [r_0 : \cdots : r_d]$ in $\mathbb{P}^d$ with homogeneous ideal W, it is immediate that $W_j^\perp = \operatorname{span}_k((r_0 y_0 + \cdots + r_d y_d)^j)$. If $W^{(1)}, \dots, W^{(1+d+e)}$ are the ideals of the points of $V(\widetilde{J})$, then we have

$$\widetilde{J}_j^\perp = \left(\bigcap_{i=1}^{1+d+e} W^{(i)} \right)_j^\perp \supseteq (W^{(1)})_j^\perp + \cdots + (W^{(1+d+e)})_j^\perp.$$

Hence (1). For (2), let $l_j := y_0 + a_1^{(j)} y_1 + \dots a_d^{(j)} y_d$ and observe that

$$q_j = y_0^2 - l_j^2 + \sum_{i=1}^d a_i^{(j)} \left((y_0 + y_i)^2 - y_0^2 \right).$$

Thus, the q_j belong to $\widetilde{J}_2^\perp \cap k[y_1, \dots, y_d]$. To prove that the q_j are linearly independent it suffices to show that all squares of linear forms in part (1) are linearly independent. Note that the squares of linear forms are precisely the points in the image of the second Veronese embedding of $\mathbb{P}^d$ via the complete linear system $|2H|$. This image does not lie in any hyperplane of $\mathbb{P}^{\binom{d+2}{2}-1}$, and therefore the squares of a generic set of $m \le \binom{d+2}{2}$ linear forms in d variables are linearly independent. Since $e \le \binom{d+1}{2}$, it then follows that the q_j are linearly independent. $\square$

Proof of Proposition 3.2. Part (1) follows from Lemma 3.3. For part (2), note that for any translation T, any $G \in GL(d)$, and any polynomial $g \in S$, we have

$$\operatorname{in}_{(1,\dots,1)}(G(T(g)) = G(\operatorname{in}_{(1,\dots,1)}(g)).$$

Thus we may assume that $V(J)$ contains the origin p_0 and the d canonical basis vectors $p_1, \dots, p_d$. Moreover, $V(J)$ contains e additional points p_{d+j} whose coordinates we label $(a_1^{(j)}, \dots, a_d^{(j)})$, $1 \le j \le e$.

Now let $h_2 \in \operatorname{in}_{(1,\dots,1)}(J)_2$ and let $g = h_2 + h_1 + h_0 \in J$ with $\deg(h_i) = i$. Define $\tilde{g} = h + x_0 h_1 + x_0^2 h_2 \in \widetilde{J}$. For any $\phi \in \widetilde{J}_2^\perp \cap k[y_1, \dots, y_n]$ we have $\phi h_2 = \phi(\tilde{g}) = 0$. Therefore

$$\widetilde{J}_2^\perp \cap k[y_1, \dots, y_n] \subseteq \operatorname{in}_{(1,\dots,1)}(J)_2^\perp$$

Applying Lemma 3.3, we see that $\langle q_1, \dots, q_e \rangle \subseteq \operatorname{in}_{(1,\dots,1)}(J)_2^\perp$. Part (2) of the proposition follows since the q_j are linearly independent and the right hand side has dimension e. $\square$

Using Proposition 3.1, we now estimate the dimension of the locus of smoothable ideals in $\operatorname{Gr}(3, S_2^*)$. This is an important ingredient in the proof of Theorem 1.4. We briefly review a coordinate system for R_n^d introduced in [Hai98, (2.21)]. Suppose M is a monomial ideal of colength n with standard monomials λ, and suppose that J is an ideal such that $V(J)$

consists of n distinct points $a^{(1)}, \dots, a^{(n)}$ with coordinates $a_i^{(j)}$ for $1 \le i \le d$. Fix an order $(m_1, \dots, m_n)$ on the set λ and define $\Delta_\lambda = \det\left([m_i(a^{(j)})]_{i,j}\right)$. If $J \in U_\lambda$, then we can express the coordinates $c^m_{m'}$ in terms of the $a_i^{(j)}$ using Cramer's rule as:

$$c^m_{m'} = \frac{\Delta_{\lambda - m' + m}}{\Delta_\lambda}$$

where $\lambda - m' + m$ is the ordered set of monomials obtained from λ by replacing m' with m. Glueing over the various U_λ, these quotients determine a rational map $\Delta : (\mathbb{A}^d)^n \dashrightarrow R^d_n$ which is regular when the points $a^{(j)}$ are all distinct.

Lemma 3.4. *If $4 \le d \le 11$, then $R^d_{d+4} \cap \mathrm{Gr}(3, S_2^*)$ has codimension at most $\binom{d-2}{2}$ in $\mathrm{Gr}(3, S_2^*)$ and codimension at most 12 in R^d_{d+4}.*

Proof. We have a rational map $g := \pi \circ \Delta : (\mathbb{A}^d)^n \dashrightarrow \mathrm{Gr}(e, S_2^*)$ where π is the map introduced in Proposition 3.1. Let Y be the domain of definition of g. If $q \in Y$, then the dimension of every component of the fiber $Y_{g(q)}$ is at least $\dim(Y) - \dim(g(Y))$. It follows that for any $q \in Y$ we have

$$\dim(g(Y)) \ge \dim(Y) - \dim(T_q Y_{g(q)}).$$

This inequality allows the computation of explicit lower bounds for the dimension of the locus of smoothable $(1, d, 3)$-ideals for small values of d. Computing $\dim(T_q Y_{g(q)})$ in Macaulay2 [GS] with $k = \mathbb{Q}$ yields the following table:

d	n	$\le \dim(g(Y))$	$\dim(Gr(3, S_2^*))$	$\binom{d-2}{2}$
4	8	20	21	1
5	9	33	36	3
6	10	48	54	6
7	11	65	75	10
8	12	84	99	15
9	13	105	126	21
10	14	128	156	28
11	15	153	189	36

Thus, for $4 \le d \le 11$ the codimension of the intersection $R^d_{d+4} \cap \mathrm{Gr}(3, S_2^*)$ in $\mathrm{Gr}(3, S_2^*)$ is at most $\binom{d-2}{2}$. By semicontinuity of fiber dimensions, the lower bound obtained by computation over $\mathbb{Q}$ holds over a field of any charactersitic. Finally, the last statement of the proposition follows since the dimension of R^d_{d+4} is $d(d+4)$. ☐

4. κ-vectors and Betti numbers

In this section, we elaborate on the definition of κ-vector, and we discuss some of its elementary properties. We compute κ_0 and κ_1 of a generic $(1, d, e)$-ideal and of a generic *smoothable* $(1, d, e)$-ideal. These computations will be used in the proofs of Theorems 1.2, 1.3 and 1.4. We also reinterpret the entries of the κ-vector of I as the graded Betti numbers of a certain module constructed from I. This interpretation reveals surprising dependencies among the entries of the κ-vector.

Since $\mathrm{char}(k) \ne 2$, we will think of elements of S_2^* as symmetric linear transformations from S_1 to S_1^*. Let $I \subseteq S$ be a homogeneous ideal with Hilbert function $(1, d, e)$ and choose a basis $\mathbf{A} = (A_1, \dots, A_e)$ of $I_2^\perp$.

Definition 4.1. *For $0 \leq j \leq e-1$ let $\psi_j(\mathbf{A})$ be the linear map from $S_1 \otimes \wedge^j I_2^\perp$ to $S_1^* \otimes \wedge^{j+1} I_2^\perp$ given by:*

$$\psi_j(\mathbf{A})(u \otimes E) = u \otimes (E \wedge \mathbf{A}) := \sum_{i=1}^{e} A_i(u) \otimes (E \wedge A_i).$$

We define the κ-vector $\kappa(I) = (\kappa_0(I), \ldots, \kappa_{e-1}(I))$ by

$$\kappa_j(I) := \operatorname{rank}(\psi_j(\mathbf{A})).$$

Note that A_i is playing different roles on the two sides of the tensor. On the left-hand side, $A_i \in \operatorname{Hom}(S_1, S_1^*)$, so that $A_i(u) \in S_1^*$. On the right-hand side, A_i is an element of the vector space $I_2^\perp$, so that $E \wedge A_i \in \wedge^{j+1} I_2^\perp$.

Lemma 4.2. *The κ-vector has the following properties:*

1. *(1) The κ-vector $\kappa(I)$ does not depend on the choice of basis of $I_2^\perp$ and is invariant under linear changes of coordinates on S.*
2. *(2) Each κ_j is lower semicontinuous on $\operatorname{Gr}(e, S_2^*)$.*
3. *(3) The κ-vector is symmetric: $\kappa_j = \kappa_{e-j-1}$ for every $j \leq \lfloor \frac{e}{2} \rfloor$.*
4. *(4) Let $e = 3 \mod 4$ and let $e = 4f+3$. Assume further that $\binom{4f+3}{2f+1}$ and d are odd. Then $\kappa_{2f+1}(I) < d\binom{4f+3}{2f+1}$.*

Proof. (1) Suppose that $\gamma \in GL(e)$ is the change of basis from $\mathbf{A}$ to some other basis $\gamma(\mathbf{A})$. Set $\Gamma_j := Id_d \otimes (\wedge^j \gamma)$. Note that:

$$\Gamma_j^{-1} \psi_j(\gamma(\mathbf{A})) \Gamma_{j+1} = \psi_j(\mathbf{A})$$

It follows immediately that $\kappa_j(I)$ does not depend on our choice of basis of $I_2^\perp$.

Next let $\beta \in GL(d)$ and let $B_j := \beta \otimes \wedge^j Id_e$. Then we have:

$$B_j^t \psi_i(\mathbf{A}) B_{j+1} = \psi_i(\beta^t \mathbf{A} \beta)$$

Thus $\kappa_i(I)$ is invariant under the $GL(d)$-action.

(2) For a fixed sequence $\vec{s} \in \mathbb{N}^e$, the locus $\{I \in \operatorname{Gr}(e, S_2^*) | \kappa(I) \leq \vec{s}\}$ is cut out by determinantal conditions.

(3) The matrices A_i are symmetric, and thus $\psi_j(\mathbf{A})^t = \pm \psi_{e-1-j}(\mathbf{A})$.

(4) The conditions guarantee that $\psi_{2f+1}(\mathbf{A})$ is a skew-symmetric matrix of odd size, and hence it cannot have full rank. □

We now compute κ_0 and κ_1 for some $(1, d, e)$-ideals.

Proposition 4.3. *Let $e \geq 3$, let I be a generic $(1, d, e)$-ideal, and let I' be a generic smoothable $(1, d, e)$-ideal.*

1. *(1) (**Generic case**)*
   - *$\kappa_0(I) = d$.*
   - *$\kappa_1(I) = ed$ unless $e = 3$ and d is odd, in which case $\kappa_1(I) = 3d - 1$.*
2. *(2) (**Generic smoothable case**) '*
   - *$\kappa_0(I') = d$.*
   - *If $d \geq \binom{e}{2}$, then $\kappa_1(I') \leq (e-1)d + \binom{e}{2}$. Further, if $e = 3$ then $\kappa_1(I') = 2d + 2$.*
   - *$\kappa_i(I') \leq (d+e)\binom{e-1}{i}$ for $i = 1, \ldots, e-1$.*

Proof. Throughout this proof we will use the isomorphism $S_1 \to S_1^*$ given by $x_i \mapsto y_i$ so that the compositions $\psi_{j+1} \circ \psi_j$ are well defined. This allows us to define a sequence of vector spaces $\mathbb{K}(\mathbf{A}) := (S_1^* \otimes \bigwedge^\bullet (I_2)^\perp, \psi_\bullet)$.

(1) Since I is generic, we may assume that $I_2^\perp$ contains a quadric A_e of full rank. Therefore $\kappa_0(I) = d$. Moreover, $\psi_j(\mathbf{A})$ has the block form

$$\psi_j^s = \begin{pmatrix} -\psi_{j-1}^{s-1} & A_s \otimes Id_e \\ 0 & \psi_j^{s-1} \end{pmatrix}$$

where ψ_j^t is $\psi_j(A_1, \ldots, A_t)$. Since the κ-vector is independent of the coordinates chosen in S we may assume that A_e equals the identity matrix I so that

$$\begin{pmatrix} -\psi_{j-1}^{e-1} & A_e \otimes Id_e \\ 0 & \psi_j^{e-1} \end{pmatrix} \begin{pmatrix} Id_e & 0 \\ \psi_{j-1}^{e-1} & Id_e \end{pmatrix} = \begin{pmatrix} 0 & Id_e \\ \psi_j^{e-1}\psi_{j-1}^{e-1} & \psi_j^{e-1} \end{pmatrix}$$

and thus for every $j \geq 1$,

$$\kappa_j(I) = d\binom{e-1}{j} + \operatorname{rank}\left(\psi_j^{e-1} \circ \psi_{j-1}^{e-1}\right)$$

For $j = 1$ we distinguish two cases. If $e = 3$, then $\psi_1^{e-1}\psi_0^{e-1}$ coincides with the commutator $[A_2, A_1]$. If we choose A_2 to be generic antidiagonal and A_1 to be generic diagonal, then $[A_2, A_1]$ has rank d (resp. $d-1$) if d is even (resp. odd). On the other hand, if $e \geq 4$, then the composition $\psi_1^{e-1}\psi_0^{e-1}$ is a $d \times \binom{e}{2} d$ block matrix containing $[A_2, A_1]A_1 \wedge A_2 + [A_3, A_2]A_2 \wedge A_3$. Choosing A_2 and A_1 as in the case $e = 3$, we see that this matrix has full rank d when d is even, or rank at least $d-1$, when d is odd. In the odd case, all entries of $[A_2, A_1]$ in the middle row are zero. For generic A_3, the commutator $[A_3, A_2]$ will have nonzero entries in the middle row. Hence, $[A_2, A_1]A_1 \wedge A_2 + [A_3, A_2]A_2 \wedge A_3$ has full rank d.

(2) From Proposition 3.2 part (2), we see that $(I_2')^\perp$ contains a quadric of full rank. Hence $\kappa_0(I') = d$. Now, let $d \geq \binom{e}{2}$. Since I' is generic smoothable, Proposition 3.1 implies that we may choose an ideal J of distinct points such that $I' = \operatorname{in}_{(1,\ldots,1)}(J)$. We will show that

$$\kappa_1(I') \leq (e-1)d + \binom{e}{2}.$$

By symmetry of the κ-vector, it suffices to show that the above inequality holds for κ_{e-2}. Lemma 3.2 implies that, after possibly changing coordinates on S_1, the subspace $(I')_2^\perp$ has a basis $A_1, \ldots, A_e$ consisting of matrices $A_i = E_i - D_i$ where D_i is the diagonal matrix with entries $(D_i)_{kk} = a_k^{(i)}$ and E_i is the rank one matrix $\vec{a}_{(i)}\vec{a}_{(i)}^t$, where $\vec{a}_{(i)} = (a_1^{(i)}, \ldots a_d^{(i)})$. Moreover, $\psi_{e-2}(\mathbf{A}) = \psi_{e-2}(\mathbf{E}) - \psi_{e-2}(\mathbf{D})$. As a result

$$\kappa_{e-2}(\mathbf{A}) \leq \dim\left(\operatorname{Im}(\psi_{e-2}(\mathbf{D})) + \operatorname{Im}(\psi_{e-2}(\mathbf{E}))\right) \tag{1}$$

$$= \kappa_{e-2}(\mathbf{D}) + \kappa_{e-2}(\mathbf{E}) - \dim(\mathrm{W})$$

where $W = \operatorname{Im}(\psi_{e-2}(\mathbf{D})) \cap \operatorname{Im}(\psi_{e-2}(\mathbf{E}))$. To prove the theorem we will estimate the terms appearing in the right hand side.

First note that $\psi_{e-2}(\mathbf{E})$ is a block matrix of the form

$$\begin{array}{c} \\ \widehat{1} \\ \widehat{2} \\ \widehat{3} \\ \vdots \\ \widehat{e} \end{array} \begin{array}{c} \begin{array}{cccc} \widehat{1,2} & \widehat{1,3} & \dots & \widehat{e-1,e} \end{array} \\ \begin{pmatrix} \pm E_2 & \pm E_3 & \dots & \vdots \\ \pm E_1 & 0 & \dots & \vdots \\ 0 & \pm E_1 & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \pm E_{e-1} \end{pmatrix} \end{array}$$

having $\binom{e}{2}$ block columns of rank at most two. Hence $\kappa_{e-2}(\mathbf{E}) \leq \min\{de, 2\binom{e}{2}\}$. On the other hand the D_i are diagonal matrices and thus $\mathbb{K}(\mathbf{D})$ is isomorphic to the direct sum of e copies of the reduced cohomology chain complex of the d-simplex. It follows that $\mathbb{K}(\mathbf{D})$ is exact and moreover, since the a_i are generic, that $\kappa_{e-2}(\mathbf{D}) = d(e-1)$. Now, let η be the matrix obtained from $\psi_{e-2}(\mathbf{E})$ by extracting the first 2 columns from each block of $\psi_{e-2}(\mathbf{E})$. Note that η is injective and that $\mathrm{Im}(\eta) = \mathrm{Im}(\psi_{e-2}(\mathbf{E}))$. By exactness of $\mathbb{K}(\mathbf{D})$, W is isomorphic to the kernel of the composition $\psi_{e-1}(\mathbf{D}) \circ \eta$. This composition is a $d \times 2\binom{e}{2}$ matrix consisting of $\binom{e}{2}$ blocks each of which is a $d \times 2$ matrix of the form

$$\left(D_i \vec{a_{(j)}} {a_1}^{(j)} - D_j \vec{a_{(i)}} {a_1}^{(i)}, D_i \vec{a_{(j)}} {a_2}^{(j)} - D_j \vec{a_{(i)}} {a_2}^{(i)}\right).$$

Its range lies in the span of the $d \times 2\binom{e}{2}$ matrix of the form $\left(D_i \vec{a_{(j)}}, D_j \vec{a_{(i)}}\right)$. The latter matrix has rank $\min(d, \binom{e}{2})$ since $D_i \vec{a_{(j)}} = D_j \vec{a_{(i)}}$. As a result we have

$$\kappa_{e-2}(\mathbf{E}) - \dim_k W \leq \min\{de, 2\binom{e}{2}\} - \left(2\binom{e}{2} - \min\{d, \binom{e}{2}\}\right).$$

Since we assume that $d \geq \binom{e}{2}$, this simplifies to

$$\kappa_{e-2}(\mathbf{E}) - \dim_k W \leq 2\binom{e}{2} - (2\binom{e}{2} - \binom{e}{2}) = \binom{e}{2}$$

Combining this inequality with (1), we obtain the upper bound from the proposition.

Now we consider κ_1 in the case $e = 3$. Note that the upper bound given is $2d + 3$, but since $\psi_1(\mathbf{A})$ is skew-symmetric, this implies that $\kappa_1(I') \leq 2d + 2$. To verify the desired equality, we produce an example. Using notation as in Lemma 3.3, we specialize to the case $p_{d+1} = (1, \dots, 1), p_{d+2} = (1, 1, 0, \dots, 0)$ and $p_{d+3} = (0, \dots, 0, 1, 1)$. We claim that the first $2d+1$ rows of the corresponding matrix $\psi_1(\mathbf{A})$ are linearly independent. Since A_1 has rank d, the first $2d$ rows are linearly independent. Let w be the vector:

$$w := (-d+2)x_d \otimes A_1 + (-d+4)x_1 \otimes A_3 + (d-1)x_1 \otimes A_3 + \sum_{i=1}^{d-1} x_i \otimes A_1$$

The vector w belongs to the kernel of the submatrix spanned by the first $2d$ rows of $\psi_1(\mathbf{A})$, but not to the kernel of the first $2d+1$ rows. Thus $\psi_1(\mathbf{A})$ has rank at least $2d+1$; since it is skew-symmetric, it therefore has rank at least $2d+2$. This completes the proof for κ_1.

Finally, we consider the general case of $\kappa_i(I')$. We think of $(A_1, \dots, A_e)$ as an element of $k^e \otimes k^d \otimes k^d$ via the injection $\mathrm{Sym}_2(k^d) \subseteq k^d \otimes k^d$. It is clear that Definition 4.1 could be extended to any 3-tensor in $k^e \otimes k^d \otimes k^d$. Further, observe that if $x, x' \in k^e \otimes k^d \otimes k^d$ then

$\kappa_i(x+x') \leq \kappa_i(x)+\kappa_i(x')$. Proposition 3.2 part (2) implies that $\mathbf{A}$ can be written as the sum of $d+e$ pure 3-tensors in $k^e \otimes k^d \otimes k^d$. Hence, to prove the inequality for $\kappa_i(I')$, it suffices to compute $\kappa_i(x)$ in the case that x is a pure tensor. We may express any pure 3-tensor as the sequence $(A_1, 0, \dots, 0)$ where A_1 is a symmetric rank 1 matrix. It follows that $\kappa_i(x) = \binom{e-1}{i}$, which proves the claim for $\kappa_i(I')$.

□

Remark 4.4. A similar argument as in the proof of Proposition 4.3 part (1) shows that, if $e \geq 3$ and $j \leq \frac{e}{2}$, then

$$\kappa_j(I) \geq d\left(\binom{e-1}{j} + \binom{e-2}{j-1}\right)$$

We omit the proof since it will not be used in this paper, and because computer experiments indicate that κ_j is considerably larger.

Now we reinterpret the κ-vector in terms of the graded Betti numbers of a certain module. Given $I \in Gr(e, S_2^*)$ with basis $\mathbf{A} = (A_1, \dots, A_e)$, consider the $T := k[z_1, \dots, z_e]$-graded module $M(\mathbf{A})$ whose graded pieces are $M_0 = S_1$, $M_1 = S_1^*$ and $M_j = 0$ if $j \notin \{0, 1\}$. The action $z_i \cdot (u_0 + u_1)$ is defined by $A_i(u_0)$. The relationship between the Betti numbers of M and the κ-vector is summarized in the following proposition. To simplify the formulas we set $\kappa_{-1} = \kappa_e = 0$.

Proposition 4.5. *For $0 \leq i \leq e$ the graded Betti numbers of $M(\mathbf{A})$ satisfy*

$$b_{i,s} = \begin{cases} d\binom{e}{i} - \kappa_{i-1}(I), \text{ if } s = i \\ d\binom{e}{i} - \kappa_i(I), \text{ if } s = i+1 \\ 0, \text{ else} \end{cases}$$

Conversely, the components of the κ-vector of I can be expressed in terms of the Betti numbers of M as

$$\kappa_j(I) = d\binom{e}{j} - b_{j,j+1}(M).$$

Proof. Recall that $b_{i,s}(M) = \dim \mathrm{Tor}^i(M, k)_s$ [Eis05, Prop. 1.7]. The right hand side is the s-graded piece of the i-th homology of the complex $\mathbb{F} := \mathbb{K}(z_1, \dots, z_e) \otimes_T M$ obtained by tensoring the Koszul complex on $z_1, \dots, z_e$ with the T-module M. In our case the complex $\mathbb{F}$ is

$$\cdots \to \bigwedge^{e-i-1} I_2^{\perp} \otimes_k M(-i-1) \to \bigwedge^{e-i} I_2^{\perp} \otimes_k M(-i) \to \bigwedge^{e-i+1} I_2^{\perp} \otimes_k M(-i+1) \to \dots$$

and in particular the graded component of $\mathbb{F}$ in degree i is the complex:

$$\text{degree } i: \qquad \cdots \to 0 \to \bigwedge^{e-i} I_2^{\perp} \otimes_k S_1 \to \bigwedge^{e-i+1} I_2^{\perp} \otimes_k S_1^* \to 0 \to \dots$$

Similarly, the graded component of $\mathbb{F}$ in degree $i+1$ is:

$$\text{degree } i+1: \quad \cdots \to 0 \to \bigwedge^{e-i} I_2^{\perp} \otimes_k S_1 \to \bigwedge^{e-i+1} I_2^{\perp} \otimes_k S_1^* \to 0 \to \dots$$

The differentials of these complexes are $\psi_{e-i}(\mathbf{A})$ and $\psi_{e-i-1}(\mathbf{A})$ respectively. The formulas in the proposition then follow from the symmetry of the κ-vector. □

Using the notation from §2.4, Proposition 4.5 may be summarized by writing $\beta(M(\mathbf{A}))$ as

$$\begin{pmatrix} d\binom{e}{0} & d\binom{e}{1}-\kappa_0(I) & d\binom{e}{2}-\kappa_1(I) & \dots & d\binom{e}{e}-\kappa_{e-1}(I) \\ d\binom{e}{0}-\kappa_0(I) & d\binom{e}{1}-\kappa_1(I) & d\binom{e}{2}-\kappa_2(I) & \dots & d\binom{e}{e} \end{pmatrix}.$$

4.1. **Boij-Söderberg theory and κ-vectors.** Using Boij-Söderberg theory, we show that the entries of the κ-vector are interdependent. Let $m = \min\{i \geq 0 | \kappa_i < d\binom{e}{i}\}$. Let $M = M(\mathbf{A})$ be the graded module associated to some basis $\mathbf{A}$ of $I_2^{\perp}$, as defined in Proposition 4.5. The Betti diagram of M then has the following shape.

$$\begin{array}{cccccccccc} 0 & 1 & \dots & m-1 & m & \dots & e-1-m & e-m & \dots & e-1 \end{array}$$
$$\begin{pmatrix} * & * & \dots & * & * & \dots & * & - & \dots & - \\ - & - & \dots & - & * & \dots & * & * & \dots & * \end{pmatrix}$$

A $*$ represents a nonzero entry, and a $-$ represents a zero. Observe that m is the smallest integer for which $b_{m,m+1} \neq 0$.

Proposition 4.6. *With notation as above, the graded Betti numbers of M satisfy the following inequalities:*

$$b_{i,i+1}(M) \geq b_{m,m+1}(M)(i+1-m)\frac{(m+1)!(e-m)!}{(i+1)!(e-i)!}$$

for all $m \leq i \leq \lfloor \frac{e-1}{2} \rfloor$. Equivalently, the κ-vector of I satisifes the inequalities:

$$\kappa_i(I) \leq d\binom{e}{i} - \left(d\binom{e}{m} - \kappa_m(I)\right)(i+1-m)\frac{(m+1)!(e-m)!}{(i+1)!(e-i)!}$$

for all $m \leq i \leq \lfloor \frac{e-1}{2} \rfloor$.

Proof. This proof uses the terminology from the introduction of [ES07]. Let δ_m be the degree sequence $(0, 1, \dots, m-1, m+1, \dots, e+1) \in \mathbb{N}^{e+1}$ and let D be the unique pure diagram corresponding to δ_m with $b_{m,m+1}(D) = b_{m,m+1}(M)$. From the Herzog-Kühl equations [ES07, p. 2] it follows that D is the diagram:

$$\frac{b_{m,m+1}(M)}{\binom{e+1}{m+1}} \cdot \begin{pmatrix} m\binom{e+1}{0} & \dots & 1\binom{e+1}{m-1} & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & 1\binom{e+1}{m+1} & \dots & (i+1-m)\binom{e+1}{i+1} & \dots & (e+1-m)\binom{e+1}{e+1} \end{pmatrix}.$$

Since the Betti diagram of M is symmetric, the Decomposition Algorithm of [ES07] implies that the difference of diagrams $\beta(M) - D$ will be a new diagram consisting entirely of nonnegative entries. In particular, for every $i \geq m$ we have that:

$$\begin{aligned} b_{i,i+1}(M) &\geq b_{i,i+1}(D) \\ &= \frac{b_{m,m+1}(M)}{\binom{e+1}{m+1}}\left((i+1-m)\binom{e+1}{i+1}\right). \end{aligned}$$

Simplifying the right-hand side proves the first statement. The second statement then follows by applying Proposition 4.5. ☐

It would be interesting to determine all sequences which equal the κ-vector of some ideal. The previous proposition shows that many symmetric vectors in $\mathbb{N}^e$ do not occur as the κ-vector of some ideal.

Example 4.7. *Let $d = 5$ and $e = 5$, and let $I \in \mathrm{Gr}(5, S_2^*)$. If I is generic then $\kappa(I) = (5, 25, 50, 25, 5)$ and the Betti diagram of M is:*

$$\begin{pmatrix} 5 & 20 & 25 & - & - & - \\ - & - & - & 25 & 20 & 5 \end{pmatrix}.$$

Imagine, however, that we choose I so that $\kappa(I) = (5, 22, \kappa_3, 22, 5)$ for some $\kappa_3 \leq 50$. Then the Betti diagram of M looks like:

$$\begin{pmatrix} 5 & 20 & 28 & 50-\kappa_3 & 3 & - \\ - & 3 & 50-\kappa_3 & 28 & 20 & 5 \end{pmatrix}.$$

Boij-Söderberg theory implies that the entries of the following difference of diagrams must be nonnegative:

$$\begin{pmatrix} 5 & 20 & 28 & 50-\kappa_3 & 3 & - \\ - & 3 & 50-\kappa_3 & 28 & 20 & 5 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 1 & - & - & - & - & - \\ - & 15 & 40 & 45 & 24 & 4 \end{pmatrix}.$$

Proposition 4.6 implies that $50 - \kappa_3 \geq 8$, or that κ_3 is at most 42.

5. κ-CYCLES

In this section, we use the κ-vector to define $GL(S_1)$-equivariant subsets of the grassmanian $\mathrm{Gr}(e, S_2^*)$. These will be used in the proof of Theorem 1.4. Recall that the scheme $\mathrm{Gr}(e, S_2^*)$ is equivariant with respect to the $GL(S_1)$-action on $\Lambda^e \operatorname{Sym}_2(S_1^*)$. More explicitly, if $\mathbf{A} \in \mathrm{Gr}(e, S_2^*)$ is an e-dimensional vector space, then $g \in GL(S_1)$ acts by $g \cdot \mathbf{A} \mapsto g\mathbf{A}g^t$.

Definition 5.1. *Let $\vec{s} = (s_0, \ldots, s_{e-1})$ be a sequence of positive integers and let $I \in \mathrm{Gr}(e, S_2^*)$. We say that $\kappa(I) \leq \vec{s}$ if $\kappa_i(I) \leq s_i$ for all i. The κ-cycle $\Xi(\vec{s})$ is defined as the closed subset of the Grassmannian $\mathrm{Gr}(e, S_2^*)$ given by*

$$\Xi(\vec{s}) = \{I \in \mathrm{Gr}(e, S_2^*) : \kappa(I) \leq \vec{s}\}.$$

Example 5.2. *Let $e = 1$. Then for any $d' \leq d$, the κ-cycle $\Xi(d')$ corresponds to the determinantal variety of symmetric $d \times d$-matrices of rank $\leq d'$.*

Lemma 4.2 implies that each κ-cycle is equivariant under the $GL(S_1)$-action. These κ-cycles play an important role in describing the intersections between components of Hilbert schemes of points. More specifically, Proposition 4.3 part (2) shows that every smoothable $(1, d, e)$-ideal belongs to the κ-cycle $\Xi(d, 2d+2, d)$. This leads us to investigate the geometry of κ-cycles of the form $\Xi(d, 2d+2, d)$.

Definition 5.3. *A vector space of quadrics $V \in Gr(e, S_2^*)$ is purely singular if for every $A \in V$, $\mathrm{rank}(A) < d$.*

Note that $\det(A)$ defines a hypersurface in $\operatorname{Spec} k[a_{ij}] = \mathbb{A}^{\binom{d+1}{2}}$. Let $P \subseteq Gr(3, S_2^*)$ be the locus of purely singular vector spaces. Then $P \subseteq \mathrm{Gr}(3, S_2^*)$ is the Fano variety of 3-planes through the origin contained in the hypersurface $V(\det(A))$.

Proposition 5.4. *Let $\operatorname{char} k = 0$. The locus $\Xi(d, 2d+2, d) - P$ is an irreducible subset of $\mathrm{Gr}(3, S_2^*)$ of codimension $\binom{d-2}{2}$.*

Proof. Let
$$T = \mathbb{A}^{\binom{d+1}{2}} \times \mathbb{A}^{\binom{d+1}{2}}$$
parametrize pairs (B, C) of symmetric $d \times d$ matrices. Note that $T = \operatorname{Spec}(k[b_{ij}, c_{ij}])$ for $1 \le i \le j \le d$, where we think of b_{ij} as the entries of B and c_{ij} as the entries of C. We have a surjective rational map
$$p : T \times GL(S_1) \dashrightarrow Gr(3, S_2^*) - P$$
which sends
$$((B, C), g) \mapsto \operatorname{span}\{gIdg^t, gBg^t, gCg^t\} \in \operatorname{Gr}(3, S_2^*)$$
Let $X \subseteq T$ be the determinantal subscheme defined by $\operatorname{rank}(BC - CB) \le 2$.

We first claim that X is an integral subscheme of codimension $\binom{d-2}{2}$ in T. If N is a skew-symmetric $d \times d$-matrix of variables over $\mathbb{Z}[x_{ij}]$ for $1 \le i < j \le d$, then the ideal J generated by the $(2d + 4) \times (2d + 4)$-pfaffians of N is generically perfect (c.f. [KL79] or [DCEP82, p. 53] and [BV88, Prop. 4.1]). Furthermore, [BV88, Thms. 3.9 and 3.13] show that the same statement holds if we specialize the entries of the matrix to a regular sequence. Finally, [BPV90, Thm. 3.1] shows that the entries of the matrix $BC - CB$ are a regular sequence on $k[b_{ij}, c_{ij}]$; it follows that X is an integral subscheme of codimension $\binom{d-2}{2}$.

Let p' be the restriction of p to $X \times GL(S_1)$. We claim that the map $p' : X \times GL(S_1) \dashrightarrow \operatorname{Gr}(e, S_2^*)$ surjects onto the set $\Xi(d, 2d + 2, d) - P$. To see this, note that by performing row and column operations on the matrix $\psi_1(Id, B, C)$, it follows that $\kappa_1(Id, B, C) \le 2d + 2$ if and only if the rank of $BC - CB \le 2$. This shows that $\Xi(d, 2d + 2, d) - P$ is irreducible.

By semicontinuity, the dimension of a general fiber of p' is at least the dimension of a general fiber of p. Hence:
$$\dim(X \times GL(S_1)) - \dim(\Xi(d, 2d + 2, d) \setminus P) \ge \dim(T \times GL(S_1)) - \dim(\operatorname{Gr}(e, S_2^*)),$$
and it follows that
$$\operatorname{codim}(\Xi(d, 2d + 2, d) - P, \operatorname{Gr}(e, S_2^*)) \ge \operatorname{codim}(X, T) = \binom{d-2}{2}$$
On the other hand, since $\Xi(d, 2d+2, d)$ is locally cut out by the $(2d+4)\times(2d+4)$-Pfaffians of a $3d\times 3d$ matrix, the codimension is at most $\binom{d-2}{2}$. Hence $\Xi(d, 2d+2, d)-P$ is an irreducible subset of codimension $\binom{d-2}{2}$, as claimed. □

We now wish to extend the result of the previous lemma from the open set $\operatorname{Gr}(3, S_2^*) \setminus P$ to the whole grassmanian. We do this by showing that the codimension of P is sufficiently large.

Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be three $d \times d$ matrices of indeterminates. Let u, v, w be new indeterminates and let $M := uA+vB+wC$. If we specialize A, B, and C to be symmetric matrices, then coefficients in $k[a_{ij}, b_{ij}, c_{ij}]$ of the determinant of M define an ideal L which cuts out the preimage of P under the rational map $\operatorname{Spec} k[a_{ij}, b_{ij}, c_{ij}] \dashrightarrow \operatorname{Gr}(3, S_2^*)$.

In order to produce the desired upper bound for the dimension of $V(L)$, we choose a monomial ordering $\preceq$ and find an ideal $L' \subseteq \operatorname{in}_{\preceq}(L)$ of high codimension. We introduce some notation. Let $\preceq$ be the revlex order determined by any total ordering on the variables such that $c_{ij} \prec b_{k,l} \prec a_{m,n}$ and such that $i + j > k + s$ implies $h_{i,j} \preceq h_{k,s}$ for $h \in \{a, b, c\}$. Let α, β, γ be nonnegative integers such that $\alpha + \beta + \gamma = d$. Let $f_{\alpha,\beta,\gamma} \in k[a_{ij}, b_{ij}, c_{ij}]$ be the coefficient of the monomial $u^\alpha v^\beta w^\gamma$ in the determinant $D := \det(M)$.

We say that a sequence $\mathcal{S} \in \{0,1,2,3\}^d$ is of type (α, β, γ) if it contains α 1's, β 2's, and γ 3's. Given a sequence $\mathcal{S}$ of type (α,β,γ) we build a $d \times d$ matrix $M_{\mathcal{S}}$ whose i-th column is the i-th column of uA, vB, wC or M depending on whether $\mathcal{S}_i$ is $1,2,3$ or 0 respectively.

Lemma 5.5. *With notation as above, we have:*

(1) *Among all monomials appearing in the $r \times r$ minors of A, the unique $\preceq$-maximal monomial is $m^* := \prod_{i=1}^r a_{i,r+1-i}$.*

(2) *For nonnegative integers α, β, γ,*

$$\frac{\partial^{\alpha+\beta+\gamma} D}{\partial u^\alpha \partial v^\beta \partial w^\gamma} = \sum_{\mathcal{S} \text{ of type } (\alpha,\beta,\gamma)} \det(M_{\mathcal{S}}).$$

(3) *For nonnegative integers α, β, γ such that $\alpha+\beta+\gamma = d$, the unique maximal monomial appearing in $f_{\alpha,\beta,\gamma}$ is*

$$\prod_{i=1}^{\gamma} c_{i,\gamma+1-i} \prod_{j=1}^{\beta} b_{\gamma+j,\beta+\gamma+1-j} \prod_{k=1}^{\alpha} a_{\beta+\gamma+k,\alpha+\beta+\gamma+1-k}.$$

Proof. (1) A monomial m appears in an $r \times r$ minor of A if and only if there exist subsets $I, I' \subseteq \{1,\ldots,d\}$ of cardinality r, and a bijection $\sigma\colon I \to I'$, such that $m = \prod_{i \in I} a_{i,\sigma(i)}$. Among such monomials m, we will have $m \prec m^*$ if m contains at least one variable of the form $a_{i,\sigma(i)}$ with $i + \sigma(i) > r+1$. For every such monomial m we have: $\sum_{i\in I} i + \sigma(i) = \sum_{i \in I} i + \sum_{i' \in I'} i' \geq r(r+1)$. Hence, if $I \neq \{1,\ldots,r\}$ or $I' \neq \{1,\ldots,r\}$, then the previous inequality is strict. In this case, m contains a variable $a_{i,j}$ with $i + j > r+1$. If on the other hand $I = I' = \{1,\ldots,r\}$ and $i + \sigma(i) \leq r+1$ for all i, then the bijection σ must be $\sigma(i) = r+1-i$.

(2) follows by induction on $\alpha+\beta+\gamma$ by the well known fact that any partial derivative of the determinant of a matrix can be expressed as a sum of determinants of the matrices obtained by taking partial derivatives of the columns one at a time.

(3) From part (2), it follows that every monomial appearing in $f_{\alpha,\beta,\gamma}$ can be written as

$$m = \prod_{i \in I_1} a_{i,\sigma(i)} \prod_{j \in I_2} b_{j,\sigma(j)} \prod_{k \in I_3} c_{k,\sigma(k)}$$

where I_1, I_2, I_3 is a set partition of $\{1,\ldots,d\}$ with cardinalitites α, β, γ, and where σ is a permutation in S_d. Since $\preceq$ is reverse lexicographic, we can maximize parts c, b and a independently and in that order. The statement then follows by part (1). $\square$

Corollary 5.6. *The κ-cycle $\Xi(d, 2d+2, d)$ is irreducible of codimension $\binom{d-2}{2}$ when $4 \leq d \leq 8$.*

Proof. Since $\Xi(d,2d+2,d)$ is cut out by an ideal generated by the $2d+4 \times 2d+4$-Pfaffians of a $3d \times 3d$ skew-symmetric matrix, we have that every component of $\Xi(d,2d+2,d)$ has codimension at most $\binom{d-2}{2}$. If we can show that $\operatorname{codim}(P, \operatorname{Gr}(3, S_2^*)) > \binom{d-2}{2}$, then it will follow from Proposition 5.4 that $\Xi(d, 2d+d, d)$ is irreducible. Consider the rational map

$$p : \operatorname{Spec} k[a_{ij}, b_{ij}, c_{ij}] \dashrightarrow \operatorname{Gr}(3, S_2^*)$$

which sends a triple of symmetric matrices to their span. Since the fibers of p have constant dimension, we have that $\operatorname{codim}(\overline{p^{-1}(P)})$ equals the codimension of P in the grassmanian.

Specializing Lemma 5.5 to the case of symmetric matrices, we obtain explicit formulas for producing monomials in the $\preceq$-initial ideal of the ideal defining $\overline{p^{-1}(P)}$. Implementing these formulas in Macaulay2 yields the following lower bounds for the codimension of P:

d	$\binom{d-2}{2}$	$\operatorname{codim}(P, \operatorname{Gr}(3, S_2^*))$
4	1	≥ 9
5	3	≥ 11
6	6	≥ 13
7	10	≥ 15
8	15	≥ 17

□

6. Proofs of Theorems 1.2, 1.3, and 1.4

We are now prepared to prove Theorems 1.2, 1.3 and 1.4.

Proof of Theorem 1.2. Let I' be a generic smoothable $(1, d, e)$-ideal. Proposition 4.3 part (2) implies that $\kappa(I')$ satisfies conditions (*) and (**). Since the κ-vector is lower semicontinuous on $\operatorname{Gr}(e, S_2^*)$, it follows that these conditions are necessary for the smoothability of I. □

Example 6.1. *The criteria of Theorem 1.2 allows for the explicit construction of nonsmoothable ideals, and the proof of Proposition 4.3 suggests a method for constructing examples. For instance, let $d = 15$ and let $q_1 = \sum_{i=1}^{15} y_i^2, q_2 = \sum_{i=1}^{15} iy_i^2$ and $q_3 = \sum_{i=1}^{7} y_i y_{15-i}$. Let I be the $(1, 15, 3)$-ideal with $I_2^\perp = \langle q_1, q_2, q_3\rangle$. Then $\kappa_1(I) = 44$ and thus I is nonsmoothable.*

Remark 6.2. The bounds from the above theorem for κ_1 of a smoothable ideal give a partial response to Problem 18.40 of [MS05]. In particular, let $U \subseteq \operatorname{Gr}(e, S_2^*)$ be some open affine defined by inverting one of the Plücker coordinates. Then we may define a map of free modules:

$$\Psi_1 : \wedge^1(\mathcal{O}_U)^e \to \wedge^2(\mathcal{O}_U)^e$$

which specializes to $\psi_1(I)$ for any $I \in U$. Let $f = (e-1)d + \binom{e}{2}$ and let F be any $(f+1) \times (f+1)$-minor of Ψ_1. Note that F vanishes on $R^d_{1+d+e} \cap U$ since $\kappa_1(I) \leq f$ for any $I \in R^d_{1+d+e} \cap U$ by Proposition 4.3 part (2). Let g the rational map $(\mathbb{A}^d)^n \dashrightarrow \operatorname{Gr}(e, S_2^*)$ as in Lemma 3.4. The pullback $g^*(F)$ then induces an algebraic relation among the determinants Δ_λ for each F. It would be interesting to give a more invariant description of these relations among the Δ_λ, and to give a combinatorial proof of the corresponding algebraic identities.

Proof of Theorem 1.3. Let I define a minimal degree subscheme of $\mathbb{A}^d$ which is not smoothable and which cannot be embedded in $\mathbb{A}^{d-1}$. We may assume that S/I is local. If the degree of I is strictly less than $d+3$, then the Hilbert function of its associated graded ring is either $(1, d), (1, d, 1), (1, d, 1, 1)$ or $(1, d, 2)$. Propositions 4.12 and 4.13 of [CEVV] show that all such ideals are smoothable. Now let I have degree $d+4$. If the Hilbert function of the associated graded ring of I is not $(1, d, 3)$, then it must be either $(1, d, 1, 1, 1)$ or $(1, d, 2, 1)$. Propositions 4.12, 4.14 and 4.15 of [CEVV] show that all such ideals are smoothable as well. Hence it only remains to consider ideals whose associated graded ring has Hilbert function $(1, d, 3)$. Every such ideal is homogeneous. Theorem 1.2 implies that a generic $(1, d, 3)$-ideal is not smoothable for $d \geq 4$. □

Proof of Theorem 1.4. Let $\mathcal{Z}$ be the locus of smoothable $(1,d,3)$-ideals. By Theorem 1.2, $\mathcal{Z} \subseteq \Xi(d,2d+2,d)$. By Lemma 3.4, when $d \leq 11$ the set $\mathcal{Z}$ of smoothable $(1,d,3)$-ideals has codimension at most $\binom{d-2}{2}$ in $\mathrm{Gr}(3,S_2^*)$. Hence, for $d \leq 11$, the equality $\mathcal{Z} = \Xi(d,2d+2,d)$ holds if this κ-cycle is irreducible and has codimension $\binom{d-2}{2}$ in $\mathrm{Gr}(3,S_2^*)$. Part (1) then follows by Corollary 5.6 and part (2) follows from Proposition 5.4.

For the last statement of the theorem, we let $r : \mathbb{A}^d \times \mathrm{Gr}(3,S_2^*) \to H^d_{d+4}$ act by translations. Restricting the map r to $\mathbb{A}^d \times \mathcal{Z}$ gives an injection into R^d_{d+4}, which is not surjective. Hence the dimension of $\mathcal{Z}$ is strictly less than $d(d+4) - d = d^2 + 3d$. On the other hand, if $d \geq 12$ then $\dim \Xi(d,2d+2,d) \geq d^2 + 3d$. It follows that $\mathcal{Z} \subsetneq \Xi(d,2d+2,d)$. Thus there exist smoothable and nonsmoothable ideals with $\kappa(I') = (d,2d+2,d)$, and therefore knowledge of the κ-vector is not sufficient for deciding smoothability when $d \geq 12$. □

Remark 6.3. For $d = 9,10,11$, we do not know if the hypothesis that $I_2^{\perp}$ contains a nonsingular quadric is necessary for the conclusion in part (2) of Theorem 1.4.

7. Examples and Proof of Proposition 1.6

In this section, we explain how κ-vectors were used to produce the examples from the introduction. We first show that κ-vectors provide information about deformations of 0-schemes beyond smoothability.

We say that an ideal J in $k[x_1,\ldots,x_d]$ is a $(1,d',e)^{+d-d'}$-ideal if it can be written as $J = J' \cap J''$ where J'' is a homogeneous ideal with Hilbert function $(1,d',e)$ and J' is the ideal of a collection of $d-d'$-distinct points, none of which is the origin. We refer to J'' as the $(1,d',e)$-component of J.

Theorem 7.1. *Let I be a $(1,d,e)$-ideal in $k[x_1,\ldots,x_d]$.*

(1) If $\kappa_0(I) \leq d'$, then I deforms to a $(1,d',e)^{+d-d'}$-ideal whose $(1,d',e)$-component J'' satisfies $\kappa(J'') = \kappa(I)$.

(2) If I deforms to a $(1,d',e)^{+d-d'}$-ideal J with $(1,d',e)$-component J'', then $\kappa(I) \leq \kappa(J'')$.

Proof. Assume that $\kappa_0(I) = d_0 \leq d'$. Then, up to $GL(d)$-action, $(I_2)^{\perp}$ is spanned by e quadrics in $k[y_1,\ldots,y_{d_0}]$. As a result, the ideal

$$J'' := (x_{d'+1},\ldots,x_d) + I$$

is a $(1,d',e)$-ideal and $\kappa(J'') = \kappa(I)$ since $J''_2 = I_2$. Let J' be the ideal of a collection of $d-d'$ points $p_{d'+1},\ldots,p_d$ in $\mathbb{A}^d$ with the property that their last $d-d'$ coordinates are linearly independent, and define $J := J' \cap J''$. Our choice of J' ensures that, for any $c_{d'+1},\ldots,c_d \in k$, there exists a linear form $\ell(x_{d'+1},\ldots,x_d)$ with $\ell(p_{d'+i}) = c_{d'+i}$ for all i. Hence, for any $q \in I$, there exists a linear form ℓ_q such that $q + \ell_q \in J$. Since J contains no linear form, it follows that $\mathrm{in}_{(1,\ldots,1)}(J) \subseteq I$. Both ideals coincide since they have the same colength. Thus I deforms to J, proving (1).

For part (2), let Z be the subset of the Hilbert scheme consisting of $(1,d',e)^{+d-d'}$-ideals Q whose $(1,d',e)$-component Q'' satisfies $\kappa(Q'') \leq \kappa(J'')$. By assumption, $I \in \overline{Z}$. Consider the map $\pi : Z \dashrightarrow Gr(e,S_2^*)$ given by $J \mapsto \mathrm{in}_{(1,\ldots,1)}(J)$. If $Q \in Z$ does not contain a linear form, then we have $\kappa(\pi(Q)) = \kappa(Q'')$. Hence $\kappa(\pi(Q)) \leq \kappa(J'')$ for all points in $\pi(Z)$. Since $I \in \overline{\pi(Z)}$, it follows from semicontinuity of the κ-vector that $\kappa(I) \leq \kappa(J'')$. □

Proof of Proposition 1.6. We first prove the proposition in the case that $n = 15$ and $d = 11$. Hence, we must produce a subset $Z \subseteq R^{11}_{15}$ of codimension 1 such that every point of Z is singular in the Hilbert scheme. Consider the action $r : \mathbb{A}^{11} \times H^{11}_{15} \to H^{11}_{15}$ by translations and let Y be the image of $\mathrm{Gr}(3, S_2^*) \times \mathbb{A}^{11}$ in H^{11}_{15} under this action. Note that Y is not contained in the smoothable component. We define $Z := R^{11}_{15} \cap Y$. By Lemma 3.4, it follows that

$$\dim Z = 153 + \dim \mathbb{A}^{11} = 164 = \dim R^{11}_{15} - 1.$$

Since Z belongs to the intersection of two components of H^{11}_{15}, it follows that every point of Z is singular in the Hilbert scheme. Note also that every element of Z has a deformation to a smooth 0-scheme and has a deformation to a nonsmoothable 0-scheme.

We now let $n \geq 15$ and $d \geq 11$ and define the family $\widetilde{Z}_{n,d}$ where an element of $\widetilde{Z}_{n,d}$ corresponds to a 0-scheme $\Gamma = \Gamma_1 \sqcup \Gamma_2 \subseteq \mathbb{A}^d$ of degree n such that Γ_1 is abstractly isomorphic to some element of Z and such that Γ_2 is the union of $n - 15$ distinct points. By [Art76, p. 4], it follows that any abstract deformation of a 0-scheme can be a lifted to an embedded deformation of the 0-scheme in any embedding. We may then conclude that every element of $\widetilde{Z}_{n,d}$ has a deformation to a smooth 0-scheme and has a deformation to a nonsmoothable 0-scheme. In particular, $\widetilde{Z}_{n,d}$ lies on the smoothable component and at least one other component of the Hilbert scheme, and hence every point in $\widetilde{Z}_{n,d}$ is singular in the Hilbert scheme.

It remains to show that $\dim \widetilde{Z}_{n,d} = \dim R^d_n - 1 = nd - 1$. First, we choose an $11-$dimensional vector subspace $V \subseteq \langle x_1, \dots, x_d \rangle$, and then we choose an ideal $I \subseteq Sym(V) \cong k[z_1, \dots, z_{12}]$ such that I belongs to the family Z from above and such that I is supported at the origin. Next, we choose a basis $l_1, \dots, l_{d-11}$ of $\langle x_1, \dots, x_d \rangle / V$ and a basis p_1, p_2, p_3 of $Sym_2(V)/I_2$, and we choose parameters $\lambda_{i,j} \in k$ for $i = 1, \dots, d - 11$ and $j = 1, 2, 3$. By considering the ideal generated by $I + \langle l_i + \sum_{j=1}^{3} \lambda_{i,j} p_j \rangle$, we have parametrized the possible choices for Γ_1 supported at the origin. The dimension of this family is

$$11(d - 11) + (\dim(Z) - 11) + 3(d - 11) = 14d - 1$$

To parametrize all possible choices for Γ, may also translate Γ_1 anywhere in $\mathbb{A}^d$, and we may choose any generic $n - 15$ points in $\mathbb{A}^d$ for Γ_2. This yields:

$$\dim \widetilde{Z}_{n,d} = (14d - 1) + d + (n - 15)d = nd - 1$$

as desired. □

Example 1.7 part (1). Let I be a generic $(1, d, e)$-ideal and let I' be a generic smoothable $(1, d, e)$-ideal. If $e > 3$ and $d > \binom{e}{2}$, then by Proposition 4.3 part (1), we conclude that

$$\kappa_1(I') \leq (e - 1)d + \binom{e}{2} < ed = \kappa_1(I).$$

Since κ_1 is lower-semicontinuous, it follows that whenever $\kappa_1(I') > (e - 1)d + \binom{e}{2}$, the ideal I' is not smoothable. For $e = 3$, there are two cases to consider. If d is even, then $\kappa_1(I') = 2d + 3 < 3d = \kappa_1(I)$ since $d \geq 4$. If d is odd, then $\kappa_1(I') = 2d + 3 < 3d - 1 = \kappa_1(I)$ since $d \geq 5$. □

Example 1.7 part (2). We wish to show that the Hilbert scheme of 11 points in $\mathbb{A}^7$ has two components whose intersection is *not* contained in the smoothable component. Let Y_1 be the irreducible component of H^7_{11} containing the set of $(1,6,3)^{+1}$ ideals. We claim that the dimension of Y_1 is at most 77 and that Y_1 is not the smoothable component.

Consider the $(1,6,3)$-ideal $J'' \subseteq \mathbb{Q}[x_1,\dots,x_6]$ defined by

$$(J''_2)^\perp = \langle y_1^2+y_2^2+\cdots+y_6^2, y_1^2+2y_2^2+3y_3^2+5y_4^2+7y_5^2+11y_6^2, y_1y_6+y_2y_5+y_3y_4\rangle.$$

Let J' be the ideal of the point $(0,0,0,0,0,0,1)$ in $\mathbb{A}^7$ and let $J := (J''+(x_7)) \cap J'$ in $\mathbb{Q}[x_1,\dots,x_7]$. Note that J is a $(1,6,3)^{+1}$-ideal. Using Macaulay2 [GS], we compute that the tangent space dimension of J is 77. Hence, the component Y_1 has dimension at most 77. Since $\kappa(J'') = (6,18,6)$, it follows from Theorem 1.2 and [Art76, p. 4] that J is not smoothable, and thus that Y_1 is not the radical component.

Let Y_2 be the irreducible component containing the set of $(1,7,3)$-ideals. We claim that Y_2 is neither Y_1 nor the smoothable component. This follows immediately from the fact that

$$\dim Y_2 \geq \dim \mathrm{Gr}(3, S_2^*) \times \mathbb{A}^7 = 82 > 77 = \dim R^7_{11} \geq \dim Y_1.$$

We now show that the ideal $I := \mathrm{in}_{(1,\dots,1)}(J)$ is a nonsmoothable ideal which belongs to $Y_1 \cap Y_2$. Note that I belongs to Y_1 because it is a degeneration of J, and that I belongs to Y_2 since it is a $(1,7,3)$-ideal. The proof of Theorem 7.1 implies that $\kappa(I) = \kappa(J'') = (6,18,6)$. Since $18 > 2\cdot 7+2$, Theorem 1.2 implies that I is not smoothable. □

Example 1.7 part (3). We must show that some deformations of I are determined by the κ-vector according to the following table:

I deforms into a ...	if and only if ...
union of 9 points	$\kappa(I) \leq (5,12,5)$
$(1,4,3)^{+1}$-ideal	$\kappa(I) \leq (4,12,4)$
smoothable $(1,4,3)^{+1}$-ideal	$\kappa(I) \leq (4,10,4)$

The first line of the table follows from Theorem 1.4. The second line of the table follows from Theorem 7.1. We now consider the last line of the table. By [Art76, p. 4] and Theorem 1.4, a $(1,4,3)^{+1}$ ideal J is smoothable if and only if its $(1,4,3)$-component J'' has $\kappa(J'') \leq (4,10,4)$. The last line of the table then follows from Theorem 7.1. □

Remark 7.2 (Generalized κ-vector). The notion of κ-vector can be extended to subspaces of polynomials of any degree. In particular, let $V \in \mathrm{Gr}(e, S_m^*)$ be a subspace with basis $\mathbf{f} = \langle f_1,\dots,f_e\rangle$. Define the κ-matrix $\kappa^{\mathrm{mat}}(V) := (\kappa_{i,j}(V))$, where $\kappa_{i,j}(V)$ is the rank of the linear map:

$$S_i \otimes \bigwedge^j V \xrightarrow{\wedge \mathbf{f}} S_{i-m} \otimes \bigwedge^{j+1} V$$

and S_k is defined to be S^*_{-k} if $k < 0$. It would be interesting to know if this generalized numerical invariant induces further nontrivial obstructions for deformations of homogeneous ideals.

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Department of Mathematics, University of California, Berkeley, CA 94720, USA
E-mail address: derman@math.berkeley.edu
URL: http://math.berkeley.edu/~derman

Department of Mathematics, University of California, Berkeley, CA 94720, USA
E-mail address: velasco@math.berkeley.edu
URL: http://math.berkeley.edu/~velasco