OPEN QUESTIONS RELATED TO THE BOIJ-SÖDERBERG THEOREMS

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Notation: B_{mod} is the semigroup ring generated by the Betti diagrams of actual R-modules., and B_{int} is the set of lattice points coming from positive rational combinations of pure diagrams.

1. The integral structure of Betti diagrams

(1) In projective dimension two, what is B_{mod}? For level modules, (i.e. modules whose generators and socle are each concentrated in a single degree), it is known that B_{int} equals B_{mod} [Erm]. The simplest example in codimension two case where generators of B_{mod} have not be computed is the shape:

$$\begin{pmatrix} * & * & - \\ - & * & * \\ - & * & * \end{pmatrix}.$$

(2) Does $B_{\text{mod}} = B_{\text{int}}$ in the projective dimension two case? In the codimension two and Cohen-Macaulay case?

(3) Let a be an odd integer and consider the Betti sequence $(2, 2a+2, 2a+2, 2)$. Is this sequence realizable by a Cohen-Macualay module for each a?

(4) Let $\mathbf{d}$ be a degree sequence, and let $r_{\mathbf{d}}$ be the ray defined by the pure diagram $\pi_{\mathbf{d}}$.
   (a) The authors of [EFW] conjecture that all sufficiently large lattice points of $r_{\mathbf{d}}$ belong to B_{mod}. Is this really the case?
   (b) On the ray $r_{\mathbf{d}}$, which lattice point is the first that belongs to B_{mod}? The constructions of pure diagrams in [EFW] and [ES] place an upper bound on this number, but it is known that in many cases these bounds are far from optimal.
   (c) Can we find lower bounds for the first lattice point of $r_{\mathbf{d}}$ that belongs to B_{mod}?

(5) Let r be a nonpure ray in the cone of Betti diagrams. Do all sufficiently large lattice points of r belong to B_{mod}?

(6) Fix a Hilbert function $\vec{h}$, up to scaling. What does the subsemigroup generated by all modules with Hilbert function $\vec{h}$ look like?

(7) What subset of B_{int} has Hilbert function $\vec{h}$?

(8) Can one find a small number of obstructions to explain mysterious behaviour in resolutions? For instance, are the "ghost syzygies" of [**?**] related to a more general phenomena? By an "obstruction" we simply mean anything which prevents a diagram in B_{int} from belonging to B_{mod}.

(9) There is an example [Kun] of a virtual Betti diagram in B_{int} that belongs to B_{mod} if and only if $\mathrm{char}\, k \neq 2$. To what extent does B_{mod} depend on the characteristic?

(10) For $\vec{d} = (d_0, \ldots, d_s)$ is there a Cohen-Macaualy dimension 1 module M with pure resolution of type $\vec{d}$ and which is supported on $e(M)$ general points in $\mathbb{P}^s$?

The motivation behind this question is to build modules with pure resolutions from the mapping cones on modules with pure resolutions of a smaller codimension. For example, if we want to build a module with pure resolution:

$$\begin{pmatrix} 1 & - & - \\ - & 3 & 2 \end{pmatrix}$$

we could start by building modules M_1 and M_2 with resolutions:

$$\beta(M_1) = \begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix} \text{ and } \beta(M_2) = \begin{pmatrix} 2 & 2 \\ - & - \end{pmatrix}$$

If we could then find an injection from $M_2(-1) \to M_1$ we could let M be the cokernel. By the mapping cone, we could conclude that $\beta(M)$ equals the desired pure diagram.

(11) Can we bound the number of generators of B_{mod}, or the size of a minimal generator of B_{mod}?

References for these questions include: [BS08], [EFW], [ES], and [Erm].

2. Equivariant Resolutions

Let k a field, $R = Sym(E), E = k^n$, with the action of $GL(E)$.

(1) Let the characteristic of k equal 0. Let $\mathbf{F}$ be an equivariant acyclic complex. Is it (up to a multiple) a non-negative combination of pure complexes? (Multiple means tensoring by a representation). In other words, do there exist pure resolutions $\mathbf{G}_i$ and representations V_i and V such that:

$$\mathbf{F} \otimes V \cong \oplus_i \mathbf{G}_i \otimes V_i$$

(2) Let **char** $k = 0$. The Pieri formula tells us that $S_\lambda E \otimes S_i E = \oplus_\mu S_\mu E$. Let R be the polynomial ring. For each μ which appears in the summand, this Pieri formula induces maps:

$$\phi_{\lambda,\mu} : S_\mu E \otimes R(-i) \to S_\lambda E \otimes R$$

In [EFW] the authors resolve these maps for specific μ, but how does one resolve these maps in general?

(3) Can we find nice $\mathbb{Z}$-forms of these maps $\phi_{\lambda,\mu}$? Can we find equivariant pure resolutions in characteristic p?

The best reference for these questions is [EFW].

3. Decomposition into Pure Diagrams

(1) For a module M, write $\beta(M) = \sum_{i=1}^t q_i \pi_i$ with π_i pure diagrams. Does there exist a positive integer N and modules M_i which satisfy $\beta(M_i) = N q_i \pi_i$, and such that $M^{\oplus N}$ and $\bigoplus_i M_i^{\oplus N q_i}$ live on the same component of the Quot scheme? Is there a deformation between $M^{\oplus N}$ and $\bigoplus_i M_i^{\oplus N q_i}$?

(See the Observations section below for a computation related to this question.)

(2) If M and N are modules with the same Hilbert function (so that they represent points on the Quot scheme), do there exist m and n so that $M^{\oplus m}$ and $N^{\oplus n}$ are on the same component of the Quot scheme? Example:

$$\begin{pmatrix} 1 & - & - & - \\ - & 2 & - & - \\ - & - & 1 & - \\ - & 4 & 8 & 4 \end{pmatrix} \quad \text{vs.} \quad \begin{pmatrix} 1 & - & - & - \\ - & 2 & 1 & - \\ - & 1 & - & - \\ - & 3 & 8 & 4 \end{pmatrix}$$

The modules with these Betti diagrams live on different components of the Hilbert scheme. If a high direct sum of these two modules did exist, it would pass through the following Betti diagram:

$$2\begin{pmatrix} 1 & - & - & - \\ - & 2 & - & - \\ - & - & - & - \\ - & 3 & 8 & 4 \end{pmatrix}$$

which is realizable by a module (Boij).

(3) For a module M on the smooth component of the Quot (or Hilbert) scheme, is $M \oplus M$ still on the smooth component?

4. Pure resolutions in other settings

(1) Replace the polynomial ring and $\mathbb{P}^{n-1}$ by a Grassmanian and its homogeneous coordinate ring. Or by a toric variety and its homogeneous coordinate ring.
   (a) Are there analogous notions for pure diagrams and vector bundles with supernatural cohomology in these settings?
   (b) Are there analogues of any of the other aspects of the Boij-Söderberg theorems (spanning, simplicial structure, pairing between vector bundles and Cohen-Macaulay modules) in these settings?

(2) Suppose we consider modules over a non-standard graded ring.
   (a) Do pure resolutions exist? Do pure diagrams generate all Betti tables?
   (b) What do the Herzog-Kühl equations say in this case?
   (c) The simplest nontrivial example seems to be the case $k[x, y], \deg(x) = 2, \deg(y) = 3$.

(3) Suppose instead we chose a multi-graded ring.

(4) Consider the Betti sequences of Cohen-Macaulay modules in the non-graded case.
   (a) Let $\mathbf{b} = (b_0, \ldots, b_s)$ be sequence for which the Euler characteristic and all the partial Euler characteristics vanish. Does there exist a Cohen-Macaulay module M whose Betti nubmers equal $(b_0, \ldots, b_s)$?
   (b) More generally, what conditions are needed on $\mathbf{b}$ to ensure that there exists a Cohen-Macaulay module with Betti sequence $\mathbf{b}$?

Observations from the Conference

(1) Betti diagrams of the form:

$$\begin{pmatrix} p & 2p-q & - \\ - & 2q-p & q \end{pmatrix}$$

Then these modules satisfy the conditions of Question 1. This is because the corresponding Quot scheme is smooth and irreducible. Thus, for instance, if $I = (x, y^2)$ and $M = (R/I)^{\oplus 3}$ then

$$\beta(M) = \begin{pmatrix} 3 & 3 & - \\ - & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & - \\ - & - & 2 \end{pmatrix} + \begin{pmatrix} 2 & 3 & - \\ - & - & 1 \end{pmatrix}$$

If we choose M_1, M_2 any pure modules with Betti diagrams as in the right-hand side of the above equation, then M and $M_1 \oplus M_2$ will live on the same component of a smooth Quot scheme, and thus deform into one another.

More generally, the same holds for Betti diagrams of the shape

$$\begin{pmatrix} * & - & - \\ - & - & - \\ & \vdots & \\ - & - & - \\ - & * & - \\ - & * & - \\ - & - & - \\ & \vdots & \\ - & - & - \\ - & - & * \end{pmatrix}$$

(2) One of Eisenbud-Floystad-Weyman's constructions of modules with pure resolutions involved maps induced by Young tableux. These maps can be built up from the maps obtained by adding boxes one at a time to a Young tableux. In the case of adding one box at a time, Joanna Nilsson shows how to explicitly choose bases in order to assure that the matrix representations of these maps is quite simple.

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