Morita Equivalence

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Abstract

In this paper, we will survey some aspects of Morita Equivalence. After a short review of the original in the ring theory, we will talk about the work of Marc Rieffel on the Morita equivalence of operator algebras. The geometric aspects of the theory will be clarified in the theme of Poisson Geometry. Finally, we discuss some aspects of the theory on the Noncommutative tori as the simplest examples for Noncommutative Geometry and Deformation Quantization.

This is the first draff.

1 Introduction

It is very common that, to understand an object, people try to understand the way it acts on other objects. In another way, we can understand a mathematical object via its category of representations. It motivate the concept of Morita equivalence. In this note, we will try to view mathematics from the Morita equivalence perspective. After the introduction. we will devote the section 2 to the Morita equivalence of rings and algebras. In the section 3, we will come to the vast theory of Morita equivalence of C^*-algebras invented by M. Rieffel and its applications to representation theory. In the next section, we will try to discover the geometric aspects of the theory by relating it to Poisson geometry. Finally, some developments of

Morita equivalence in Noncommutative geometry and quantization are also reviewed.

Because of the limits of the paper, we can not touch the broad areas of Morita equivalence in groupoids, and Vonnewman algebras and the reader is encouraged to read wonderful papers [35], [8]. We also try to provide some necessary background. This is the survey paper for the class on C^*-algebra of the author, in the first year of the graduate study, at the Department of Mathematics, University of California at Berkeley.

2 Morita Equivalence of rings and algebras

To understand a mathematical object, it is very common to understand the way it acts on the other objects. Mathematically, the representation theory (or in another way, the module theory) of a group, a ring or an algebra can illuminate a lot of informations about their structure.

On the other hand, in a very broad class of categories, a classical mathematical object can be determined uniquely via the algebra of functions on it. For example, in algebraic geometry, an affine algebraic variety can be view as the spectrum of its coordinate ring, which is of finite type, non-nilpotent. In Gelfand theory, a locally compact Hausdorff space can be identified with the spectrum of a commutative C^*-algebra. In measure theory, a commutative Vonnewman algebra is nothing but the algebra of bounded Borel functions on a measure space. Thus, the theories of these kinds of algebras are equivalent to the theories of affine algebraic varieties, locally compact spaces, measure spaces.

On the other hand, the spectrum of a commutative algebra is the equivalent classes of irreducible representations which are one dimensional. Then, in these categories, two commutative algebras having the same representation theory (or in another way, the same category of module) are entirely isomorphic.

For noncommutative algebras, which can be viewed as the algebras of functions on some "quantum spaces" or on spaces with the very bad topology, the concept of "isomorphism equivalence" is too strong. We need a weaker concept but in some senses still respecting the representation theory.

Definition 2.1 *Two unital rings R and S are called Morita equivalent if and only if their categories of left modules are isomorphic.*

Generally, there is a natural way to establish a functor from the category R-Module to the category of S-Module. Let ${}_SX_R$ be a (S,R)-bimodule, i.e. an Abelian group with the action of S on the left, and of R on the right and two actions are commutative. Then for any P a R-module, ${}_SX_R \otimes_R P$ is a S-module. The conversed side is also true, with some additional conditions:

Theorem 2.2 *[38](Watt) Let T be a right exact functor from the category of R-modules to the category of S-modules, commuting to the direct sum. Then there exists a (S,R)-bimodule Q such that the functor T and* $_S Q_R \otimes \_$ *are equivalent.*

Although the proof is trivial, this result is really useful because it provides us another equivalent definition of Morita Equivalence.

Definition 2.3 *Two ring R and S are called Morita equivalent if there exist bimodules* $_R P_S$ *and* $_S Q_R$ *such that* $_R P_S \bigotimes_S {}_S Q_R \simeq_R R_R$ *and* $_S Q_R \bigotimes_R {}_R P_S \simeq_S S_S$.

As mentioned above, in the category of commutative rings, the Morita equivalence is exactly isomorphic relation. Thus, its role only emerges in the noncommutative setting. The most important example is the equivalence between R and $Mat_{n\times n}(R)$. The bimodule here will be $Mat_{n\times 1}(R)$ and $Mat_{1\times n}(R)$ with the canonical actions of R and $Mat_{n\times n}(R)$.

Thus, it means that, two Morita equivalent rings have the same categories of modules, bimodules, have the isomorphic ideal lattices. As a corollary, concepts and properties constructed on ideals and modules are the same for Morita equivalent rings. More precisely, Noetherness, Artinianness, simpleness, primitiveness, K-algebraic theory (constructed on finitely generated projective modules), cyclic cohomology, Holdchild cohomology are Morita invariances.

We can also see directly from the definition that , because $Hom_S(P, S) \simeq Q$ and $Hom_R(Q, R) \simeq R$ (S,R)-bimodule P can be determined from (R,S)-bimodule Q and vice versa. R and S can be embedded in End(Q), where Q is viewed as a Z-module and we can prove without any difficulty that R and S are full commutants of each other.

But the story does not come to an end. There are still some obstructions needed to put on R-module Q so that R and its commutant are equivalent. In [22], Morita has proved that the necessary and sufficient condition is that Q_R is a finitely generated projective generator.

3 Induced representation and strong Morita Equivalence of C^*-algebras

In this section, we investigate operator algebras from the viewpoints of representation theory and Morita equivalence. Let A be a C^*-algebra (possibly without identity). Naturally, we will look at representation theory of A on as the theory of A-Hermitian modules, i.e. a Hilbert space H and a nondegenerate $*$-representation of A on H. The collection of all Hermitian A-modules together with the corresponding of intertwining operators forms a category, which we will denote by Hermod-A. A lot of properties of the C^*-algebra

A are reflexed in this category, but some of them are lost. We can only hope to recover the C^*-algebra A from the Hermod-A with many additional conditions:

Theorem 3.1 *Let A be a C^*-algebra and H be the forgetful functor from Hermitian-A to the category of Hilbert spaces. Then the collection C of natural transformations from H to itself can be given in a natural way the structure of a W^*-algebra and this W^*-algebra is naturally isomorphic to the W^*-enveloping algebra n(A) of A.*

The result was proved by John E. Roberts, and then for the W^*-algebras by Marc Rieffel in [24]. However, there are many nonisomorphic C^*-algebras having the same enveloping W^*-algebra and they contribute to the fact that the equivalence of representation categories is weaker (or in some sense more interesting) than the isomorphic relation.

Let A and B be two C^*-algebras. As in the previous section, a (A, B)-bimodule X provide us a general tool to construct a functor from the category of right B-modules to the category of right A-modules. However , the situation here is much more complicated because we need to construct a Hibert space structure on tensor products of two Hermitian modules. In 1970s, motivated in part by the induced representation theory of locally compact groups, in [23], M. Rieffel has solved the difficulty by introducing the concept of generalized conditional expectation and then Hilbert Module. The idea is simple, but it was a big advancement in operator algebras.

Definition 3.2 *Let A be a C^*-algebra. By a (right) pre-Hilbert A-module, we mean a right A-module X equipped with a A-valued pre-inner product, that is, a A-valued sesquilinenar form $\langle , \rangle$ conjugated bilinear map in the first variable, such that*

1. $\langle x, x \rangle_A \geq 0$ *for all* $x \in X$,
2. $(\langle x, y \rangle_A)^* = \overline{\langle y, x \rangle_A}$ *for all x,y* $\in X$,
3. $\langle x, yb \rangle_A = \langle x, y \rangle_A b$ *for all x,y* $\in X$, $b \in A$.

On a pre-Hilbert A-module, we can construct a seminorm compatible with the A-valued pre-inner product by $\|x\| = \|\langle x, x \rangle_A\|^{1/2}$. As usual, we mod out all the norm-zero elements and then complete the quotient space and the byproduct is still a pre-Hilbert A-module. We call them Hilbert modules

Definition 3.3 *A Hilbert A-module is a complete pre-Hilbert A-module X under the above norm and satisfied* $\|x\| \neq 0$ *if and only if* $x = 0$.

Note that in the original paper [24], M. Rieffel called it A-rigged module. There are some examples of Morita equivalence we can meet in various places.

Example 1 *Let E be a Hermitian vector bundle (may not be locally trivial) over a compact space M. Then on the space of continuous section $\Gamma(E)$, we can construct a inner product with values on the C^*-algebra $C(M)$ by the fiber-wise inner product $\langle s_1, s_2\rangle(x) = \langle s_1(x), s_2(x)\rangle_x$. Complete $\Gamma(E)$, we gain a Hilbert C(M)-module.*

Example 2 *Let A be a C^*-algebra, B a subalgebra of A. By a conditional expectation from A to B, we mean a bounded positive projection $P : A \to B$, such that $P(b_1ab_2) = b_1P(a)b_2$ for all $a \in A$ and $b_1, b_2 \in B$. Then we can define on A a pre-Hilbert B-module structure by $\langle a_1, a_2\rangle_B = P(a_2^*a_1)$. It is one of the main step in the process of generalizing the induced representation to C^*-algebras.*

With this new tool, we can easily define the pre-Hilbert structure on the tensor product of a Hilbert A-module X and an A-Hermitian module V by $\langle x_1 \otimes v_1, x_2 \otimes v_2\rangle = \langle\langle x_1, x_2\rangle_A v_1, v_2\rangle_V$. The Hilbert space obtained by taking the completion is denoted by $X\hat{\underset{A}{\otimes}}V$.

In some aspects, Hilbert modules looks very similar to a Hilbert spaces. For example, for x in a Hilbert A-module X, like in Hilbert space, it can be proved that $\|x\|_X = \underset{y\in X, \|y\|\leq 1}{sup} \|\langle x, y\rangle_A\|$. However, there are many more things to say.

First, for F a closed Hilbert A-submodule of X, generally its bipolar is larger than F. A bounded A-module linear map T between two Hilbert A-modules X and Y may not have an bounded adjoint operator like maps between Hilbert space. In the category of Hilbert module, in order to imitate the classical theory of Hilbert space, we will only take into consideration the class of all adjointable maps between X and Y, equipped with canonical norm and denote by $\mathfrak{L}(X, Y)$. Like the wellknown algebras of bounded operators on a Hilbert space, $\mathfrak{L}(X, X)$ is also a C^*-algebra.

There is also a generalized concept of the space of compact operators for the Hilbert modules. Recall that the algebra of compact operator on Hilbert space are the closure of the space of finite rank operators, so we should define a analogous concept. Let $\theta_{x,y}$ be a operator between two A-Hilbert modules X and Y, defined by $\theta_{y,z}(x) = y\langle z, x\rangle$ for $x, z \in X, y \in Y$, which is clearly a Hilbert module version of rank one operator. We define the space of compact operators as the norm-closure of the space of "finite rank operators" generated by "those rank-one operators" $\theta_{x,y}$ and denote it by $\mathfrak{K}(X, Y)$. Now, we are ready to reach the main point in this section.

For A and B be two C^*-algebras, we define a "generalized morphism" from B to A as a right Hilbert B-module X with B-valued inner product $\langle, \rangle_B$ together with a nondegenerate *-representation of A in $\mathfrak{L}_B(X)$, in the sense that AX is dense in X with respect to the B-seminorm on X. For two generalized morphisms ${}_AX_B$ and ${}_BY_C$, we can compose them by associating on the algebraic tensor product a positive definite C-valued norm defined by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_1, \langle x_1, x_2 \rangle_B y_2 \rangle_C.$$

The completion of this space turns out to be a generalized morphism from C to A, denoted by ${}_AX_B \hat{\otimes}_B Y_C$ and called the Rieffel tensor product of ${}_AX_B$ and ${}_BY_C$. Note that, in his original paper [24] Rieffel has used another the terminology Hermitian B-rigged A-module.

Therefore, we can define a category $\mathfrak{C}$ of C^*-algebras together with the isomorphism classes of the generalized morphism between them. Two C^*-algebras are then called (strong) Morita equivalence if and only if they are isomorphic in the sense of the category $\mathfrak{C}$.

Definition 3.4 *Let A and B be C^*-algebras. Then an A-B-imprimitive bimodule is an A-B bimodule X which is equipped with an A-valued and a B-valued inner product with respect to which X is a left A-rigged space and a right B-rigged space, such that:*

1. $\langle x, y \rangle_A z = x \langle y, z \rangle_B$ *for all* $x, y, z \in X$*;*
2. $\langle ax, ax \rangle_B \leq \|a\|^2 \langle x, x \rangle_B$ *for all* $x \in X$ *and* $a \in A$*;*
3. $\langle xb, xb \rangle_A \leq \|b\|^2 \langle x, x \rangle_A$ *for all* $x \in X$ *and* $b \in B$*.*

Then, we can prove that a Morita equivalence between A and B in the $\mathfrak{C}$ category is equivalence to the existence of a A-B-imprimitive bimodule X. The set of automorphisms in the category $\mathfrak{C}$ of an object, which parameterizing the ways it can be Morita equivalent to itself, is called the Picard group.

Remark 1 *As we can guess, two isomorphic C^*-algebras are Morita equivalent. Let P be an isomorphism between C^*-algebras A and B, then we can construct the A-B-imprimitive bimodule X, with base B, acted by P(A) on the left, by B on the right and* $\langle x, y \rangle_B = x^* y$ *and* ${}_A\langle x, y \rangle = P^{-1}(xy^*)$*.*

Remark 2 *For ${}_AX_B$ an A-B-imprimitive bimodule, we can form $\bar{X}$ with the same base space but with conjugate action of A,B and get the inverted B-A-imprimitive bimodule.*

Example 3 *A Hilbert space H is naturally a $\mathfrak{K}(H)$-C imprimitive bimodule with ${}_{\mathfrak{K}(H)}\langle h, k \rangle = h \otimes \bar{k}$. Thus $\mathfrak{K}(H)$ is Morita equivalent to C, which is equivalent to a well known fact that $\mathfrak{K}(H)$ is simple. However the algebra $\mathfrak{B}(H)$ is not Morita equivalent to C as C^*-algebras, because the image of the inner product $\langle H, H \rangle_{\mathfrak{B}(H)}$ is not dense in $\mathfrak{B}(H)$. It is due to the existence of the Calkin algebra $\mathfrak{B}(H)/\mathfrak{K}(H)$ breaks the simplicity of $\mathfrak{B}(H)$. However, they are Morita equivalent as W^*-algebras.*

Example 4 *Let X be a Hilbert A-module. Then the C^*-algebras A and $\mathfrak{K}_A(X)$ are Morita equivalent.*

So an A-B-imprimitive bimodule X induce an functorial equivalence from the Hermod(B) to Hermod(A) defined by taking the Rieffel tensor product of ${}_AX_B$ with a Hermitian B-module. However, the inversion is not true, the categorical equivalence of Hermitian modules is weaker than ours and usually called the weak Morita equivalence.

We also recall that our philosophy, a noncommutative algebra can be visualized as the algebra of functions on the equivalent classes of "irreducible representations", thus approximately on its primitive spectrum, i.e. the set of the kernel of irreducible representations. So, it is very natural to anticipate that,

Theorem 3.5 *Denote $I(A), I(B)$ be the lattice of ideals of A and B. Then an A-B-imprimitive bimodule X induces a isomorphism between I(A), I(B) and the lattice of A-B-submodules of X. Further more, it is even still an isomorphism while restricting on lattices of the corresponding ideals and sub-bimodules.*

The lattice isomorphism is also called the Rieffel correspondence.

Theorem 3.6 *The Rieffel correspondence is an homeomorphism between Prim(A) and Prim(B), the primitive spectrums of A and B associated with the Hull-kernel topology.*

Definition 3.7 *By a hereditary subalgebra of a C^*-algebra A, we mean a subalgebra C such that if $a \in A$ and if $0 \leq a \leq c$ for some $c \in C$ then $a \in C$. It is called full if it is contained in no proper two sided ideal of A.*

It reveals another form of Morita equivalence: two C^*-algebras A and B are Morita equivalent if and only if can be embedded as a full hereditary subalgebras of a bigger C^*-algebra C.

We also can realized two Morita equivalent C^*-algebras and B if and only if they are stable isomorphic, i.e. $A \otimes \mathfrak{K} \simeq B \otimes \mathfrak{K}$, where $\mathfrak{K}$ is the algebra of compact operators. However, normally it is very difficult to point out directly the stable isomorphism, and therefore this results, proved by Brown- Green-Rieffel [1], seems of theoretical importance.

As mentioned above, the main motivation of M. Rieffel to introduce Morita equivalence is to understand the induced representation theory. The Mackey imprimitive theorem [20] states that, any representations of a locally compact group G obtained from an induction process from a representation of a closed subgroup H admits a system of imprimitivity. This result was firstly proved in an "purely analysis" way, by choosing a measure on G/H, then representing the algebra $C_c(G/H)$ on the induced representation. However, by making use of the Morita equivalence algebraic machine, we can avoid almost the measure theory complicated techniques, except for the very basic facts about Haar measure to construct the convolution and representations of group C^*-algebras.

From that viewpoint, a induced representation from H to G is equivalent to a covariant representation of $C^*(G, G/H)$. Thus, the Mackey imprimitive theorem became the fact that two $C^*(G, G/H)$ and $C^*(H)$ are strongly Morita equivalence, and we have the one to one correspondence between the representations of two algebras. In fact, it is a very special case of an unpublished result of Phil Green, and proved by M. Rieffel in [26].

Theorem 3.8 *Let H and K be two locally compact groups acting on a locally compact space P such that the two actions are commutative, free and wandering, in the sense that for any compact subset K of P, the set* $\{x \in G \mid xK \cap K \neq \emptyset\}$ *should be precompact in G so the two quotient spaces* P/H *and* P/K *are Hausdorff).*

Then the C^**-algebras* $C^*(K, P/H)$ *and* $C^*(H, P/K)$ *are strongly Morita equivalent.*

However, it is still difficult to find out all the unitary irreducible representation of a locally compact group because we don't know how to choose which representation from which subgroup to induce. In a special case,when the group G is an group extension, the situation becomes much easier.

Theorem 3.9 *Let G be a locally compact group and N be a normal subgroup. Let* $J \in Prim(N)$ *(the primitive spectrum) and let GJ denote the orbit of J in Prim(N) under the conjugate action of G, and let* G_J *be the stability subgroup of J. We also assume that* $\{J\}$ *and GJ is locally closed in Prim(A), and the canonical map of* G/G_J *onto GJ is a homeomorphism.*

Then the process of inducing representation from G_J *to G establishes an equivalence of the category of unitary* G_J*-modules whose restrictions to N live on* $\{J\}$ *with the category of unitary modules whose restrictions to N live on GJ. This equivalence preserves weak containment of representations.*

This result provides us a general direction to find out the unitary representations of solvable groups with a normal Abelian subgroup. There are many ways to view this result, but we can understand it by the Morita equivalence. For more details, see [27].

4 Morita Equivalence in Poisson Geometry

4.1 Some background of Poisson Geometry

In this section, we will develop the Morita equivalence theory for the category of Poisson manifold.

Let P be a smooth manifold. A Poisson structure on P is an R-bilinear Lie braket $\{,\}$on $C^\infty(P)$ satisfying the Leibnitz rule $\{f, gh\} = \{f, g\}h + g\{f, h\} = 0$ for all f,g,h in $C^\infty(P)$.

For any $f \in C^\infty(P)$, the derivative $X_f = \{f, .\}$ is called the Hamiltonian vector field of f. When $X_f = 0$ the f is called the Casimir function.

We can see that a Poisson structure is can be determined by a bivector field Π such that $\{f, g\} = \Pi(df, dg)$. The Jacobi identity is equivalent to the Schouten-Nijenhuis braket $[\Pi, \Pi]$ vanishes. In the local coordinate $(x_1, ..x_n)$, the tensor Π is determined by matrix $\Pi_{ij} = \{x_i, x_j\}$. If the matrix is invertible then we call the Poisson structure nondegenerate or symplectic.

Example 5 *Let P=R^n, and we assume the rank of the matrix Π_{ij} is constant. Then, in some coordinate, $(q_1, ..q_k, p_1, .., p_k, r_1, ..r_l)$ the Poisson structure is:*

$$\{f, g\}(x) = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

Example 6 *(Lie-Poisson structures)*
Let $\mathfrak{g}^*$ be the dual of a Lie algebra. Then we can construct a linear Poisson structure on $\mathfrak{g}^*$ by the Lie structure of $\mathfrak{g}$

$$\{f, g\}(X) = X([df, dg]).$$

Conversely, any linear Poisson structure $\{x_i, x_j\} = \sum_k c_{ij}^k x_k$ on vector space V is equivalent to a Lie algebra structure with the structure coefficients c_{ij}^k on the predual of V.

A Poisson structure Π then define a linear map from T^*P to TP . At the points that the Poisson structure is nondegenerate, the map is onto. But in general, the image just form an integrable distribution of TP. The Poisson manifold then has the structure of a singular foliation with symplectic leaves. The following splitting theorem by A. Weinstein [36] describles the local structure of any Poisson structure.

Theorem 4.1 *(Splitting theorem) Around any point x_0 in a Poisson manifold P, there always exists a local coordinate, $(q_1, ..q_k, p_1, .., p_k, r_1, ..r_l)$, with $(q, p, r)(x_0) = (0, 0, 0)$, such that*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2}\sum_{i,j=1}^{l} n_{ij}(r)\frac{\partial}{\partial r_i} \wedge \frac{\partial}{\partial r_j}$$

For example, the symplectic of a Lie-Poisson structure is nothing but its coadjoint orbits.

Definition 4.2 *A map $\psi : (P_1, \Pi_1) \to (P_2, \Pi_2$ between two Poisson manifolds is called a Poisson mapping or a Poisson morphism if for any $f, g \in C^\infty(P_2)$, one has*

$$\{f \circ \psi, g \circ \psi\}_1 = \{f, g\}_2 \circ \psi,$$

or, equivalently the tensor fields Π_1 and Π_2 are ψ-related. Or even more, we can reformulate the definitions by the relationship between the Hamiltonian vector fields $X_f = \psi_*(X_{\psi^* f})$ for any $f \in C^\infty(P_2)$.

Of course, a Poisson map may not take a complete Hamiltonian vector field to a complete Hamiltonian vector field because there may be some missing points P_2 that could not be reached by lifting trajectory in P_1.

A Poisson map will be called complete if it maps complete vector fields to complete vector fields. Note that the concept of "completeness" is only meaningful for the relation between Poisson manifolds, not for the manifolds themselves. From now, we only consider the complete map.

Remark 3 *Let (P,Π) be a connected symplectic manifold, and $U \subset P$ be an open subset. Then the inclusion of U into P is complete if and only if U is closed. As a corollary, symplectic manifolds is the minimal object in the category of Poisson manifolds with the complete maps.*

Definition 4.3 *A Lie algebroid is a vector bundle A on a smooth manifold M such that*

1. *There is a Lie algebra structure on the space of the section of the bundle.*
2. *A vector bundle homomorphism ρ from A to TM called the anchor.*
3. *$[fe_1, e_2] = f[e_1, e_2] - (\rho(e_2)f)e_1$ for all $f \in C^\infty(M), e_1, e_2 \in \Gamma(M)$*

The first example about Lie algebroids is of course $T\Sigma$ with ρ being the identity. The second example is any Lie algebra with the base M degenerating to one point, ρ trivial. And finally,$A = T^*M$ for M a Poisson Manifold is always a Lie algebroid with the anchor $\rho = P^\sharp, \rho(\alpha_i P^{ij}\partial_j)$ and the bracket $[df, dg] = d\{f, g\}$. Notice that although the bracket is define here for the exact form but we can extend it to all the form by the last axiom.

On the other hand, we define a Lie groupoid $(\mathcal{G} \longrightarrow M, s, t)$ as a smooth manifold $\mathcal{G}$ with surjective submersion s and t from $\mathcal{G}$ to M, and a smooth multiplication from $\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \times \mathcal{G} | t(h) = s(g)\}$ to $\mathcal{G}$. It can be viewed as a category such that all morphisms are invertible. The Lie groupoid is called source-simply-connected if the s-fibers are connected and simply connected. A good example of a Lie groupoid is $\Pi(\Sigma)$ the homotopy classes of curves on Σ with natural start and terminal maps.

Now, the reader who is familiar with the Lie theory can have a feeling about Lie algebroid and Lie groupoid. For any Lie groupoid, there is also an associating Lie algebroid given by $Ker(ds)|_M$ with the anchor dt and the Lie bracket induced by multiplication. A morphism between two Lie algebroids of Lie groupoids can be integrated to a Lie groupoid morphism. However, the third theorem of Lie is not true for the "oid" version. There are Lie algebroids that can not be integrated to any Lie groupoids and which ones can be integrated are called integrable. A Poisson manifold is also called integrable if the Lie algebroid T^*M is integrable as above.

There have been many attempts to understand the meaning of the concept of integrability. In 2003, Crainic and Fernandes have constructed the general conditions for Lie algebroids [6]. After 1 years, they have realized the result for T^*M where M is a Poisson manifold. The main result in [7] is

Theorem 4.4 *The following claims are equivalent:*

1. *M is integrated to a symplectic Lie groupoid*
2. *T^*M is integrable.*
3. *The Weinstein groupoid $\Sigma(M)$ is a smooth manifold*
4. *The monodromy group N_x for $x \in M$ are locally uniformly discrete.*

In [7] the integrability is proved to be equivalent to the existence of a complete symplectic realization (see the section 4.1)

4.2 Some motivation of Morita Equivalence in Poisson Geometry

. Poisson manifolds enter in the middle of the passing through classical physics to quantum physics, because they have mixed characteristic properties of the both worlds. By the splitting theorem, a Poisson manifold has the structure of a symplectic foliation, i.e. the union of phase spaces of classical mechanics systems. On the other hand, the structure of the singular foliation is very complicated, and therefore the quotient topology of the leaf space becomes very bad, namely non- Hausdorff. It also means that almost all powerful tools of classical mathematics like homotopy, homology, cohomology...when applied directly yield trivial results. Thus, a fruitful approach may be to view the very bad topological space as the spectral of some noncommutative algebras and we think that this is one of the ways Noncommutative Geometry jumping in our unifying picture.

There are many strong connections between the theory of noncommutative algebras and Poisson Geometry motivating Morita equivalence for Poisson manifolds. Our favorite viewpoint is from Harmonic Analysis, i.e. the orbit method of Kostant-Kirillov [14], [15].

By a G-homogenous Hamiltonian space, we mean a symplectic manifold P with the transitive action of a Lie group G such that the invariant vector field generated by an infinitesimal representation of $X \in$ the Lie algebra $\mathfrak{g}$ can be lifted to function f_X and the correspondence $X \to f_X$ is a Lie homomorphism. Therefore, the induced correspondence $x \to f_{\{.\}}(x)$ becomes an G-equivariant from P to $\mathfrak{g}^*$ called the moment map. All G homogeneous Hamiltonian space will be a covering space of a coadjoint orbit of G.

Therefore, there is a close relationship between G- homogenous Hamiltonian space (the phase spaces of minimal classical mechanics systems) and the coadjoint orbits of G. In another hand, the model of a quantum mechanic systems with the symmetric group G is a Hilbert space, a family of evolutions of unitary operators with the group G acting on. The philosophy of the Kostant-Kirillov orbit method is that, if we quantize in some senses the coadjoint orbits (an approximation of classical mechanics systems) we may obtain the unitary irreducible representations of G. The functor from the coadjoint orbits to irreducible representations are named the Kirillov correspondence.

$$\begin{array}{ccc} \text{Classical mechanics systems} & \overset{quantization}{\longrightarrow} & \text{Quantum mechanics systems} \\ \updownarrow & & \updownarrow \\ \text{Coadjoint orbits} & \overset{Kirillov functor}{\longrightarrow} & \text{Irreducible representations} \end{array}$$

The Kirillov correspondence is bijective when the group G is nilpotent, but failed for other cases, it requires strong modification and for many cases the correspondence is still mysterious. However, it still suggests a close relationship between the category of the symplectic with a moment maps to $g*$ (the G-homogeneous Hamiltonian spaces) and the category of irreducible representation of G, or algebraically the representation theory of the group C^*-algebra of G.

Although the connection between the geometric world, i.e. Poisson(symplectic, Hamiltonian) geometry and the algebraic world, i.e. the theory of C^*-algebras, group C^*-algebras and their quantized versions as quantum groups, quantum algebras... is not entirely illuminated now, we can still use the ideas and techniques of the algebraic world to understand the geometric world and vice versa. So, it becomes natural to develop the representation theory of Poisson manifolds, and therefore the Morita equivalence of Poisson manifold.

4.3 Representation theory of Poisson manifolds

Motivated by the previous subsection, as well as the remark in the section 5.1, we view a simply connected symplectic manifolds as fundamental object, like simple C^*-algebras of compact operators. Then, we define a (anti) representation of a Poisson manifold P is a symplectic manifold S, together with a complete (anti-)Poisson map $J : S \to P$. We also call it as a left (right) P-module, or complete (anti)symplectic realization of P.

Definition 4.5 *Given a symplectic manifold M and Poisson manifolds P_1, P_2. A symplectic dual pair is a pair of Poisson maps $J_1 : M \to P_1$ and $J_2 : M \to P_2$ such that in the Poisson algebra $C^\infty(M)$, the two subalgebras $J_1(C^\infty(P_1))$ and $J_2(C^\infty(P_2))$ are the others's commutant.*

The local meaning of the idea is that, the J_1-fiber is the symplectic orthogonal complement to the J_2-fiber and vice versa. However, in order to talk globally, it requires some obstruction on the topology of fibers.

Definition 4.6 *Two Poisson manifolds P_1 and P_2 are called Morita equivalent if there exists a symplectic manifold X together with a Poisson morphism $J_1 : X \to P_1$ and an anti-Poisson morphism $J_2 : X \to P_2$ such that $P_1 \leftarrow X \to P_2$ is a complete dual pair with connected and simply connected fibers and J_1 and J_2 are surjective submersion. X is called an equivalent bimodule.*

We can see that, the definition intimates all the properties of the Morita equivalence of rings. The geometric meaning of the Morita equivalence is reflexed as:

Theorem 4.7 *(Weinstein, [37]) There is a one to one correspondence between the symplectic leaves of P_1 and P_2.*

Now, we want to define the tensor product on modules over a symplectic manifold, like the Rieffel tensor product in C^*-algebra.

Let $J : S \to P$ be a right P-module. and $J' : S' \to P$ be a left P-module. Like in ring theory, we define the tensor product of S and S' to be the quotient of the fiber product of S and S' on P by it characteristic foliation (to remove the "zero part "). However, the terrible thing is that the quotient may not be symplectic, or even worse, not a manifold. But we can still pray and hoping that the tensor product is symplectic manifold for some good enough representations. Analogously, the space of the intertwining maps between two left Poisson modules M_1 and M_2 is then defined as $\bar{M}_1 * M_2$.

For connected symplectic manifolds, Ping Xu proved in [33] that symplectic manifolds are classified entirely up to Morita equivalence by their fundamental groups. Especially, any connected, simply connected symplectic manifold is Morita equivalent to a point with zero Poisson structure. This is compatible to our viewpoint that symplectic manifolds are very similar to the irreducible representations. For general Poisson manifolds, it is much more difficult to find out all the Morita invariances and in fact the understanding is still very limited.

Theorem 4.8 *(Ginzburg-Lu),[11] Two Morita equivalent Poisson manifolds have the isomorphic first Poisson cohomology groups.*

As a corollary of the theorem (4.7), when the leaf spaces are smooth, the two Morita equivalent Poisson manifolds are diffeomorphic. More than that, the corresponding leaves are themselves Morita equivalent, see [2] and have the isomorphic tranverse Poisson structure [31].

However, there happens that Morita equivalence of Poisson manifolds is not an equivalent relation, see [33]. It due to the fact that not all Poisson manifolds admit identity morphisms and more importantly, we don't know how to compose two morphisms, except for restricting to a smaller class of Poisson manifolds.

Theorem 4.9 *(Ping Xu, [33]) The Morita equivalence gives rise an equivalent relation in the category of integrable Poisson manifolds.*

In fact, the theory for integrable Poisson manifolds is a very special case for theory for Lie groupoids, due to the fact that their symplectic groupoids are Morita equivalent. However, because of the limited place, we will not go any further and the interested reader can cunsult in [17], [34].

5 Morita Equivalence and Noncommutative Geometry

5.1 Deformation quantization

In theis section, we will try to understand the Morita equivalence at the quantum lever. There are many ways to understand (partly) the relationship between classical and quantum mechanics as well as the quantization funtors between them.

A classical mechanic system is given by its phase space, i.e. a symplectic manifold or more general a Poisson manifold. The space of functions is then a Poisson algebra.

However, in quantum mechanic systems, we can not determine position and impulsion at the same time due to the Heisenberg uncertainty principle. Mathematically, it is necessary to replace functions on quantum space by noncommutative operators.

Quantization is therefore a process associating to each Poisson manifold P a Hilbert space H of so-called quantum states, to each classical quantity f$\in$ $C^\infty(P)$ a quantum quantity Q(f) $\in L(H)$, i.e., a linear, perhaps unbounded, normal operator which is Hermitian if f is a real-valued function such that $Q(\{f,g\}) = \frac{i}{\hbar}[Q(f), Q(g)]$, and $Q(1) = Id_H$.

There are some approaches to this problem, such as Feynman path integral quantization, pseudo differential operator quantization, geometric quantization, etc.

The main motivation of deformation quantization came from the relationship between classical and quantum mechanics.

In deformation quantization, the quantization is considered as the deformation of the structure of the Poisson algebra of classical observable via a family of associative algebras indexed by the so-called deformation parameter, rather than a change in the nature of the observables. More precisely, deformation quantization of a Poisson algebra A (in particular $C^\infty(P)$ where P is a Poisson manifold) is an associate, $C[[\hbar]]$-bilinear multiplication $*$ on A such that for f and g in A, the $*$-product satisfies

1. $f *_\hbar g = f.g + O(\hbar)$,
2. $\frac{1}{i\hbar}(f *_\hbar g) - g *_\hbar f)) = \{f, g\} + o(\hbar)$.

The conditions can be understood as a deformation of the commutative product on A along the direction of the Poisson structure, i.e. a curve on the space of the associative product structure on A, with the derivative Π.

There are many versions of the deformation quantization, depending on the Poisson algebra as well as the sense of convergence. We temporatorily take into consideration two kinds of quantizations of interest.

1. The first is the formal deformation quantization. In this case, we view $\hbar$ as a formal parameter, and then extend the quantum product into formal power series without caring about convergence. Thus the quantized product of functions can be established as an associative product on the algebra of formal series of $\hbar$ with coefficients in A.

2. The second is Rieffel strict deformation quantization. This can be viewed as the operator algebra version of the above quantization realized as a field of C^*-algebras. More precisely, for each $\hbar \in I$ an open interval containing zero, there exists an associative product $*_\hbar$, an involution $\overset{*}{\hbar}$ and a C^*-norm on A such that for $\hbar = 0$ they become the normal product, complex conjugation and supremum norm, as well as:

    (a) for any f $\in A$, the map $t \to \|f\|_\hbar$ is continuous, namely A_t is a continuous field of C^*-algebras.

    (b) for any f, g in A, $\|(f *_\hbar g - g *_\hbar f)/i\hbar - \{f, g\}\|_\hbar$ converges to zero when t tends to zero.

The Rieffel's strict deformation quantization is heavily based on the theory of C^*-algebra, and we have to take care of all the convergence. Naturally, we also can raise the question on the Morita equivalence of the strict deformation quantized C^*-algebras. However, it is too complicated in the general cases, and so only solved in some special examples. We look at the quantum tori, the simplest quantum spaces and see how our tool works out.

Let $\Pi = (\Pi ij)$ be an $n \times n$ anti-symmetric matrix. There are two main ways to define a quantum torus. The first is to define it abstractly from generators.

Definition 5.1 *We define A_Π to be the universal C^*-algebra generated by n unitaries $u_1, .., u_n$ satisfying:*

$$u_j u_k = e^{2\pi i \Pi kj} u_k u_j.$$

However, we still don't know if it exists nontrivially or not! Equivalently, we can define it directly by deforming the commutative product of functions. Let $\theta_1, .., \theta_n$ be the coordinate on the n-dimensional torus, and the Poisson structure given by $\{\theta_j, \theta_k\} = \Pi_{j,k}$. Recall that, via Fourier transformation, $C(T^n)$ is isomorphic to the Schwartz spaces of function on Z^n with convolution

$$\hat{f} * \hat{g}(n) = \sum_{k \in Z^n} \hat{f}(n)\hat{g}(n-k).$$

Now, we use the matrix θ to twist this product with a deformation constant:

$$\hat{f} *_\hbar \hat{g}(n) = \sum_{k \in Z^n} \hat{f}(n)\hat{g}(n-k)e^{-\pi i \hbar \theta_{k,n-k}}.$$

and then pull it back by the Fourier transform to get a "quantized" product in $C(T^n)$, also called the quantum torus.

Remark 4 *It can be proved that, the functions $e^{2\pi i \theta_j}$ play the role of u_j in the first definition. Then two definitions are equivalent and the noncommutative tori are examples of Rieffel's strict deformation quantization.*

Let $O(n, n\|R)$ be the group of automorphisms of $R^n \oplus R^{n*}$ preserving the canonical symmetric bilinear form $x_1x_{n+1} + x_2x_{n+2} + .. + x_nx_{2n}$. If written in the block matrix form,

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

the group $O(n, n\|R)$ acts on the space of all $n \times n$ skew symmetric matrices by

$$\Pi \to g\Pi := (A\theta + B)(C\theta + D)^{-1}.$$

The main result in [28] is that, if Π is a skew-symmetric matrix, $g \in SO(n, n \mid R)$ and $g\Pi$ is well defined then two noncommutative tori A_Π and $A_{g\Pi}$ are

Morita equivalent. However, unluckily (or luckily, in another sense) the converse that those algebras are Morita equivalent only if those matrices Π lie in the same orbits is true for dimension n = 2 and failed for dimensions $n > 2$.

However, the formal deformation quantization process may not work well for any Poisson algebra due to a lot of obstructions the associativity of the deformed product putting on higher coefficients in term of cohomology classes. The problem of the existence and classification of deformation quantization is therefore entirely nontrivial. In 1997, Maxim Kontsevich has completed his Formality conjecture [16] and implied the isomorphism between the moduli spaces of the Poisson structures and of the deformations of the associative algebra structure.

It has also been known that there is only one deformation quantization on a connected symplectic manifold, up to an equivalence[18]. For a Poisson manifold, because of the leaf space, there may be many deformation quantizations leading to many quantum algebras (in fact they form a moduli space). A very natural question may be asked that if the quantized algebras are Morita equivalent or not. This result was obtained in the symplectic case:

Theorem 5.2 *Let P be a symplectic manifold. If the Picard group $Pic(P) = H^2(P, Z)$ has no torsion then it acts freely on the equivalence classes of star product on P, and two star product are Morita equivalent if and only if they lie in the same $H^2(P, Z)$ orbit, up to symplectomorphism.*

We bring this theorem a little bit down to earth. It has been known in [18] that equivalence classes of star products on a symplectic manifold are parameterized by elements of $\frac{1}{i\hbar}[\omega] + H^2_{DR}(P)[[\hbar]]$, where ω is the symplectic structure on P, and $H^2_{DR}(P)[[\hbar]]$ is the DeRham cohomology group of P with complex coefficients. The Picard group here is the group of the isomorphism classes of complex line bundle on M, so is $H^2(P, Z)$ and acting on the star-product equivalence classes by

$$[\omega]_\hbar \to [\omega]_\hbar + 2\pi i c_1(L)$$

where $c_1(L)$ is the image of the Chern class of the line bundle L in $H^2_{DR}(P)$. There is also a version of this result for general Poisson manifolds, but it is very complicated and we encourage the reader discover it by himself in [3], [4].

5.2 Noncommutative Field theories and Morita equivalence

In quantum physics, people try to model the micro world via a concept called Quantum Field theory. On a low dimensional boundary manifold (In the 2 dimensional case, it is understood as the surface swept out in the spacetime by strings), people investigate generalized functions or sections of a bundle and an action functional on the space of fields.

Recently, ideas of Alan Connes on investigating noncommutative C^*-algebras as if they are algebras of functions have soaked in to many areas of mathematics and physics, especially in quantum field theories. The world-sheets are the surfaces wept out by particles, so they behave in a highly noncommutative ways. Recently, it has been a fashion to introduce and understand quantum field theories on spacetime with noncommutative coordinate functions and people call it noncommutative field theories.There are many ways to quantize the world sheet and spacetime. Here, if the world sheet is associated with a Poisson structure, we can quantize it in the deformation sense.

In the commutative case, we describe the matter fields as sections of some vector bundles $\pi : E \to M$. Mathematically the space of matter fields Ξ is a bimodule on $C^\infty(M)$ and $\Gamma^\infty(End(E))$. We also define a Hermitian inner product on fibers of E, to formulate the kinematics and the dynamics of the system, we need to construct a mass term, which is a Hermitian fiber metric on E. Namely, Ξ has an $C^\infty(M)$-valued inner product.

Now assume that there is a Poisson structure on the spacetime M. Then, we can quantize $C^\infty(M)$ by a star product $*_\hbar$, see section 5.1. Therefore, the projective module structures of $C^\infty(M)$ and $\Gamma^\infty(End(E))$ as well as the $C^\infty(M)$-valued inner product on $\Gamma^\infty(E)$ should also be deformed compatibly with the star product $*_\hbar$ on M. The existence of a quantization like this was proved in [5] by deforming a Hermitian projection of some free modules on $C^\infty(M)$, or using a connection ∇ to lift the Kontsevich star product on the vector bundle to construct the module structure like by Jurčo, Schupp and Wess in [13]

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