Lectures on Logarithmic Algebraic Geometry

Arthur Ogus

February 13, 2006

Contents

Chapter I

The geometry of monoids

1 Basics on monoids

1.1 Limits in the category of monoids

A *monoid* is a triple $(M, \star, e_M)$ consisting of a set M, an associative binary operation $\star$, and a two-sided identity element e_M of M. A homomorphism $\theta : M \to N$ of monoids is a function $M \to N$ such that $\theta(e_M) = e_N$ and $\theta(m \star m') = \theta(m) \star \theta(m')$ for any pair of elements m and m' of M. Note that although the element e_M is the unique two-sided identity of M, compatibility of θ with e_M is not automatic from compatibility with $\star$. We write **Mon** for the category of monoids and morphisms of monoids. All monoids we consider here will be commutative unless explicitly noted otherwise.

We will often follow the common practice of writing M or $(M, \star)$ in place of $(M, \star, e_M)$ when there seems to be no danger of confusion. Similarly, if a and b are elements of a monoid $(M, \star, e_M)$, we will often write ab (or $a + b$) for $a \star b$, and 1 (or 0) for e_M.

The most basic example of a monoid is the set $\mathbf{N}$ of natural numbers, with addition as the monoid law. If M is any monoid and $m \in M$, there is a unique monoid homomorphism $\mathbf{N} \to M$ sending 1 to m: $\mathbf{N}$ is the free monoid with generator 1. More generally, if S is any set, the set $\mathbf{N}^{(S)}$ of functions $I\colon S \to \mathbf{N}$ such that $I_s = 0$ for almost all s, endowed with pointwise addition of functions as a binary operation, is the free (commutative) monoid with basis $S \subseteq N^{(S)}$. The functor $S \mapsto \mathbf{N}^{(S)}$ is left adjoint to the forgetful functor from monoids to sets.

Arbitrary projective limits exist in the category of monoids, and their formation commutes with the forgetful functor to the category of sets. In particular, the intersection of a set of submonoids of M is again a submonoid, and hence if S is a subset of M, the intersection of all the submonoids of M containing S is the smallest submonoid of M containing S, the *submonoid of M generated by S.* If there exists a finite subset S of M which generates M, one says that M is *finitely generated* as a monoid.

Arbitrary inductive limits of monoids also exist. This will follow from the existence of direct sums and of coequalizers. Direct sums are easy to construct: the direct sum $\bigoplus M_i$ of a family $\{M_i : i \in I\}$ of monoids is the submonoid of the product $\prod_i M_i$ consisting of those elements m such that $m_i = 0$ for almost all i. The construction of coequalizers is more difficult, and we first investigate quotients in the category of monoids.

If $\theta \colon P \to M$ is a homomorphism of monoids, then the set E of pairs $(p_1, p_2) \in P \times P$ such that $\theta(p_1) = \theta(p_2)$ is an equivalence relation on P and also a submonoid of $P \times P$, and if θ is surjective, M can be recovered as the quotient of P by the equivalence relation E. A submonoid E of $P \times P$ which is also an equivalence relation on P is called a *congruence* (or *congruence relation*) on P. One checks easily that if E is a congruence relation on P, then the set P/E of equivalence classes has a unique monoid structure making the projection $P \to P/E$ a monoid morphism. Thus there is a dictionary between congruence relations on P and isomorphism classes of surjective maps of monoids $P \to P'$. The intersection of a family of congruence relations is a congruence relation, and hence it makes sense to speak of the congruence relation generated by any subset of $P \times P$. One says that a congruence relation E is *finitely generated* if there is a finite subset S of $P \times P$ which generates E as a congruence relation; this does not imply that S generates E as a monoid.

The following proposition, whose proof is immediate, summarizes the above considerations.

Proposition 1.1.1 *Let $P \to P'$ be a surjective mapping of monoids, and let $E := P \times_{P'} P \subseteq P \times P$, i.e., the equalizer of the two maps $P \times P \to P'$.*

1. *E is a congruence relation on P.*

2. *P' is the coequalizer of the two maps $E \to P$.*

Here is a useful description of the congruence relation generated by a subset of $P \times P$.

Proposition 1.1.2 *Let P be a (commutative) monoid.*

1. *An equivalence relation $E \subseteq P \times P$ is a congruence relation if and only if $(a+p, b+p) \in E$ whenever $(a,b) \in E$ and $p \in P$.*

2. *If S is a subset of $P \times P$, let $S_P := \{(a+p, b+p) : (a,b) \in S, p \in P\}$. Then the congruence relation E generated by S is the equivalence relation generated by S_P. Explicitly, E is the union of the diagonal with the set of pairs (x,y) for which there exists a finite sequence $(s_0, \ldots s_n)$ with $s_0 = x$ and $s_n = y$ such that for every $i > 0$, either (s_{i-1}, s_i) or (s_i, s_{i-1}) belongs to S_P.*

Proof: Suppose that an equivalence relation E is closed under addition by elements of the diagonal of $P \times P$ and that (a,b) and $(c,d) \in E$. Then $(a+c, b+c)$ and $(c+b, d+b) \in E$, and since P is commutative and E is transitive, $(a+c, b+d) \in E$. Since E contains the diagonal, the identity element $(0,0)$ of $P \times P$ belongs to E, so E is a submonoid of $P \times P$, hence a congruence relation. Conversely, if E is a congruence relation, then for any $p \in P$, $(p,p) \in E$, and hence if $(a,b) \in E$, $(a+p, b+p) \in E$. This proves (1). For (2), let E denote the congruence relation generated by S and E' the equivalence relation generated by S_P; evidently $E' \subseteq E$. It follows from the associative law that S_P is closed under addition by elements of the diagonal of $P \times P$. Hence if $(s_0, \ldots s_n)$ is a sequence such that (s_{i-1}, s_i) or $(s_i, s_{i-1}) \in S_P$ for all $i > 0$, then $(s_0 + p, \ldots s_n + p)$ shares the same property. Thus if $(x,y) \in E'$ and $p \in P$, then $(x+p, y+p) \in E'$. Then it follows from (1) that E' is a congruence relation, and so $E' = E$. □

Remark 1.1.3 If Q is an abelian group and $E \subseteq Q \times Q$ is a congruence relation on Q, then the image of E under the homomorphism $h\colon Q \oplus Q \to Q$ sending (q_1, q_2) to $q_2 - q_1$ is a subgroup K of Q, and $E = h^{-1}(K)$. Conversely the inverse image under h of any subgroup of Q is a congruence on Q. This simply makes explicit the familiar correspondence between quotients of Q, subgroups of Q, and congruence relations on Q.

If u and v are two morphisms of monoids $Q \to P$, one can construct the coequalizer of u and v as the quotient of P by the congruence relation on P

generated by the set of pairs $(u(q), v(q))$ for $q \in Q$. In general, a diagram of monoids

$$\begin{tikzcd} Q \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & P \arrow[r, "w"] & R \end{tikzcd}$$

is called *exact* if w is the coequalizer of u and v. The existence of arbitrary inductive limits follows from the existence of direct sums and coequalizers of pairs of morphisms by a standard construction.

A *presentation* of a monoid M is an exact diagram

$$\begin{tikzcd} L_1 \arrow[r, shift left] \arrow[r, shift right] & L_0 \arrow[r] & M \end{tikzcd}$$

with L_0 and L_1 free. It is equivalent to the data of a map from a set I to M whose image generates M and a map from a set J to $\mathbf{N}^{(I)} \times \mathbf{N}^{(I)}$ whose image generates the congruence relation on $N^{(I)}$ defined by the surjective monoid map $\mathbf{N}^{(I)} \to M$ corresponding to the set map $I \to M$. The monoid M is said to be *of finite presentation* if it admits a presentation as above with L_0 and L_1 free of finite type. We shall see in (2.1.9) that in fact every finitely generated monoid is of finite presentation.

The *amalgamated sum* $Q_1 \xrightarrow{v_1} Q \xleftarrow{v_2} Q_2$ of a pair of monoid morphisms $u_i \colon P \to Q_i$, often denoted simply by $Q_1 \oplus_P Q_2$, is the inductive limit of the diagram $Q_1 \xleftarrow{u_1} P \xrightarrow{u_2} Q_2$. That is, the pair (v_1, v_2) universally makes the diagram

$$\begin{tikzcd} P \arrow[r, "u_1"] \arrow[d, "u_2"'] & Q_1 \arrow[d, "v_1"] \\ Q_2 \arrow[r, "v_2"] & Q \end{tikzcd}$$

commute, and can be viewed as the pushout of u_1 along u_2 or the pushout of u_2 along u_1. It can also be viewed as the coequalizer of the two maps $(u_1, 0)$ and $(0, u_2)$ from P to $Q_1 \oplus Q_2$. As the following proposition shows, the calculation of Q is considerably simplified if one of the monoids in question is a group (see (4.3.2) for a generalization).

Proposition 1.1.4 *Let $u_i \colon P \to Q_i$ be a pair of monoid morphisms, let Q be their amalgamated sum, and let E be the congruence relation on $Q_1 \oplus Q_2$ given by the natural map $Q_1 \oplus Q_2 \to Q$.*

1. *Let E' be the set of pairs $((q_1, q_2), (q_1', q_2'))$ of elements of $Q_1 \oplus Q_2$ such that there exist a and b in P with $q_1 + u_1(b) = q_1' + u_1(a)$ and $q_2 + u_2(a) = q_2' + u_2(b)$. Then E' is a congruence relation on $Q_1 \oplus Q_2$ containing E, and if any of P, Q_1, or Q_2 is a group, then $E = E'$.*
2. *If P is a group, then two elements of $Q_1 \oplus Q_2$ are congruent modulo E if and only if they lie in the same orbit of the action of P on $Q_1 \oplus Q_2$ defined by $p(q_1, q_2) = (q_1 + u_1(p), q_2 + u_2(-p))$.*
3. *If P and Q_i are groups, then so is $Q_1 \oplus_P Q_2$, which is in fact just the coproduct in the category of abelian groups.*

Proof: If $q_1 + u_1(b) = q_1' + u_1(a)$ and $q_2 + u_2(a) = q_2' + u_2(b)$, we shall say that "$(a, b)$ links (q_1, q_2) and (q_1', q_2')." The set E' is evidently symmetric and reflexive. To prove the transitivity one checks immediately that if (a, b) links (q_1, q_2) and (q_1', q_2') and (a', b') links (q_1', q_2') and (q_1'', q_2''), then $(a + a', b + b')$ links (q_1, q_2) and (q_1'', q_2''). Moreover, if (a, b) links (q_1, q_2) and (q_1', q_2') then for any $(\tilde{q}_1, \tilde{q}_2) \in Q_1 \oplus Q_2$, (a, b) links $(q_1 + \tilde{q}_1, q_2 + \tilde{q}_2)$ and $(q_1' + \tilde{q}_1, q_2' + \tilde{q}_2)$. Then by (1.1.2) E' is a congruence relation on $Q_1 \oplus Q_2$. Furthermore, if $p \in P$, $(p, 0)$ links $(u_1(p), 0)$ and $(0, u_2(p))$, and since E is the congruence relation generated by such pairs, $E \subseteq E'$. If P or either Q_i is a group, then $v := v_i \circ u_i$ factors through the group Q^* of invertible elements of Q. If (a, b) links (q_1, q_2) and (q_1', q_2'), we find that

$$v_1(q_1) + v_2(q_2) + v(a + b) = v_1(q_1') + v_2(q_2') + v(a + b),$$

and since $v(a + b) \in Q^*$, it follows that

$$v_1(q_1) + v_2(q_2) = v_1(q_1') + v_2(q_2').$$

Thus $E' \subseteq E$. This proves (1), and (2) and (3) are immediate consequences. □

Example 1.1.5 If we take $Q_2 = 0$ in 1.1.4 one obtains the *cokernel* of the morphism $u_1 \colon P \to Q_1$, or, equivalently, the coequalizer of u_1 and the zero mapping $P \to Q_1$. If P is a submonoid of Q_1, one writes $Q_1 \to Q_1/P$ for this cokernel, and it follows from (1.1.4) that two elements q and q' of Q_1 have the same image in Q_1/P if and only if there exist p and p' in P such that $q + p = q' + p'$. If P' is a submonoid of Q_1 containing P, then P'/P is a submonoid of Q_1/P, and the natural map $(Q_1/P)/(P'/P) \to Q_1/P'$ is an isomorphism.

If S is a set, then the set of functions from S to itself forms a (not necessarily commutative) monoid $\operatorname{End}(S)$ under composition. If Q is a monoid, an *action of* Q *on* S is a morphism of monoids θ from Q to $\operatorname{End}(S)$. In this context we often write the monoid law on Q multiplicatively, and if $q \in Q$ and $s \in S$, qs for $\theta(q)(s)$. A Q*-set* is a set endowed with an action of Q, and $\mathbf{Ens}_Q$ will denote the category of Q-sets, with the evident notion of morphism. If S is a Q-set and $s \in S$, the image of the map $Q \to S$ sending q to qs is the minimal Q-invariant subset of S containing s, called the *trajectory* of s in S.

A *basis* for a Q-set (S, ρ) is a map of sets $i\colon T \to S$ such that the induced map $Q \times T \to S\colon (q,t) \mapsto \rho(q)i(t)$ is bijective; if such a basis exists, we say that (S, ρ) is a *free* Q*-set*. A free Q-set with basis $T \to S$ satisfies the usual universal property of a free object: to give a map of Q-sets $(S, \rho) \to (S', \rho')$ is the same as to give a map of sets $T \to S'$. If T is any set and if $Q \times T$ is endowed with the action ρ defined by $\rho(q')(q,t) = (q'q, t)$, then the map $T \to Q \times T$ sending t to $(1, t)$ is a basis. Thus the functor taking a set T to the free Q-set $Q \times T$ is left adjoint to the forgetful functor from the category of Q-sets to the category of sets. If G is a group and S is a G-set, then S has a basis as a G-set if and only if the action is free in the sense that $gs = s$ implies $g = 1$, but this equivalence is not true for monoids in general.

The category $\mathbf{Ens}_Q$ of Q-sets admits arbitrary projective limits, and their formation commutes with the forgetful functor to the category of sets, since the forgetful functor $\mathbf{Ens}_Q \to \mathbf{Ens}$ has a left adjoint. In particular, if S and T are Q-sets, then Q acts on $S \times T$ by $q(s,t) := (qs, qt)$, and this action makes $S \times T$ the product of S and T in $\mathbf{Ens}_Q$.

Inductive limits in the $\mathbf{Ens}_Q$ also exist. The direct sum of a family $S_i :$ $i \in I$ is just the disjoint union, with the evident Q-action. To understand the construction of quotients in the category $\mathbf{Ens}_Q$, note that if $\pi\colon S \to T$ is a surjective map of Q-sets, then the corresponding equivalence relation $E \subseteq S \times S$ is a Q-subset of $S \times S$; such an equivalence relation is called a *congruence relation* on S. Conversely, if E is any congruence relation on S, then there is a unique Q-set structure on S/E such that the projection $S \to S/E$ is a morphism of Q-sets. When $S = Q$ acting regularly on itself, the notion of a congruence relation on Q as a monoid coincides with the notion of a congruence relation as a Q-set, thanks to (1.1.2). Furthermore, the analog of (2) of (1.1.2) holds for Q-sets, and in particular the equivalence relation generated by a subset of $S \times S$ which is stable under the diagonal action of Q is already a congruence relation. If u and v are two morphisms

$S' \to S$, the coequalizer of u and v is the quotient of S by the congruence relation generated by $\{(u(s'), v(s')) : s' \in S'\}$.

Suppose that S, T, and W are Q-sets. A *Q-bimorphism* $S \times T \to W$ is by definition a function $\beta\colon S \times T \to W$ such that $\beta(qs, t) = \beta(s, qt) = q\beta(s, t)$ for any $(s, t) \in S \times T$ and $q \in Q$. The *tensor product of S and T* is the universal Q-bimorphism $S \times T \to S \otimes_Q T$. To construct it, begin by regarding $S \times T$ as a Q-set via its action on S: $q(s, t) := (qs, t)$, and consider the equivalence relation R on $S \times T$ generated by the set of pairs

$$((qs, t), (s, qt)) \in (S \times T) \times (S \times T) \text{ for } q \in Q, s \in S, t \in T.$$

Note that this set of pairs is stable under the action of Q, since if $q' \in Q$, and if $s' := q's$, then $((q'qs, t), (q's, qt)) = ((qs', t), (s', qt))$. It follows that the equivalence relation R is a congruence relation. Then the projection $\pi\colon S \times T \to (S \times T)/R$ is a Q-bimorphism and satisfies the universal mapping property of the tensor product. If Q is a (commutative) group, then $S \times_Q T$ can be constructed in the usual way as the orbit space of the action of Q on $S \times T$ given by $q(s, t) := (qs, q^{-1}t)$.

Suppose that $\theta\colon Q \to P$ is a monoid homomorphism. Then θ defines an action of Q on P by $qp := \theta(q)p$. If T is a Q-set, the tensor product $P \otimes_Q T$ has a natural action of P, with $p(p' \otimes t) = (pp' \otimes t)$, and the map $T \to P \otimes_Q T$ sending t to $1 \otimes t$ is a morphism of Q-sets over the homomorphism θ. If R is the Q-set defined by a monoid homomorphism $Q \to R$, then $(p \otimes r)(p' \otimes r') = (pp' \otimes rr')$ is a monoid structure on $P \otimes_Q R$ for which the natural maps $P \to P \otimes_Q R$ and $R \to P \otimes_Q R$ are homomorphisms. It can be checked that this monoid structure makes $P \otimes_Q R$ into the amalgamated sum of P and R along Q.

Definition 1.1.6 *Let Q be a monoid and let S be a Q-set. The transporter of S is the category $\mathcal{T}_Q S$ whose objects are the elements of S, and for which the morphisms from an object s to an object t are the elements q of Q such that $qs = t$, with composition defined from the multiplication law of Q. The transporter of a monoid Q is the transporter of Q regarded as a Q-set, and is denoted simply by $\mathcal{T} Q$.*

Recall from [1, I,2.7]. that a category is said to be *filtering* if it satisfies the following conditions:

1. For any diagram of the form

$$\begin{tikzcd} s \arrow[r, "u_1"] \arrow[d, "u_2"'] & t_1 \\ t_2 & \end{tikzcd}$$

   there exist morphisms $v_1\colon t_1 \to t$ and $v_2\colon t_2 \to t$ such that $v_1 u_1 = v_2 u_2$.

2. For any diagram

$$\begin{tikzcd} s \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & t, \end{tikzcd}$$

   there exists a morphism $w\colon t \to t'$ such that $w \circ u = w \circ v$.

3. The category is (nonempty and) connected, *i.e.*, any two objects can be joined by a chain of arrows (in either direction).

The transporter category of any Q-set S satisfies (1), and the transporter category of Q is filtering.

Associated with the category $\mathcal{T}_Q S$ is a partially ordered set which is worthwhile making explicit.

Definition 1.1.7 *Let Q be a monoid and S a Q-set. If s and t are elements of S, we write $s \leq t$ if there exists a $q \in Q$ such that $qs = t$, and $s \sim t$ if $s \leq t$ and $t \leq s$.*

It is clear that if $s \leq t$ and $t \leq w$, then $s \leq w$, and that for every $s \in S$, $s \leq s$. Thus the relation $\leq$ defines a pre-ordering on S. The relation $\sim$ is a congruence relation on S, and the relation $\leq$ on $S/\sim$ is a partial ordering. We shall use this notion especially when $S = Q$ with the regular representation. Since $\sim$ is a congruence relation, it follows from 1.1.2 that $Q/\sim$ inherits a monoid structure.

1.2 Integral, fine, and saturated monoids

If M is any commutative monoid, there is a universal morphism λ_M from M to a group M^{gp}. That is, M^{gp} is a group, $\lambda_M\colon M \to M^{gp}$ is a homomorphism of monoids, and any morphism from M to a group factors uniquely through λ_M. Thus, the functor $M \mapsto M^{gp}$ is the left adjoint of the inclusion functor from the category of groups to the category of monoids; since it has a right adjoint, it automatically commutes with the formation of direct limits. In fact, M^{gp} can be identified with the cokernel 1.1.5 of $M \times M$ by the diagonal, and λ_M with the composite of $(\mathrm{id}_M, 0)$ and the projection $M \times M \to M \times M/\Delta_M$. One can also construct M^{gp} as the set of equivalence classes of pairs (x, y) of elements of M for which (x, y) is equivalent to (x', y') if and only if there exists $z \in M$ such that $x + y' + z = x' + y + z$. The explicit description of the equivalence relation in 1.1.5 shows that the two constructions are in fact the same. One writes $x - y$ for the equivalence class containing (x, y), and $(x - y) + (x' - y') := (x + x') - (y + y')$.

If M is a monoid, let M^* denote the set of all $m \in M$ such that there exists an $n \in M$ such that $m + n = 0$. Then M^* forms a submonoid of M. It is in fact a subgroup—the largest subgroup of M. We call it the *group of units* of M; it acts naturally on M by translation. One says that M is *quasi-integral* if this action is free, *i.e.*, if whenever $u \in M^*$ and $x \in M$, $u + x = x$ implies that $u = 0$. If G is any subgroup of M, the orbit space M/G can be identified with the quotient M/G the category of monoids 1.1.5. In particular,we write $\overline{M}$ for M/M^*. If M is quasi-integral, the map $M \to \overline{M}$ makes M an M^*-torsor over $\overline{M}$. A monoid M is called *sharp* if 0 is its only unit. For any monoid M, the quotient $\overline{M}$ is sharp, and the map $M \to \overline{M}$ is the universal map from M to a sharp monoid.

A monoid M is called *integral* if the cancellation law holds, *i.e.*, if $x + y = x' + y$ implies that $x = x'$. Evidently any integral monoid is quasi-integral. The universal map $\lambda_M \colon M \to M^{gp}$ is injective if and only if M is integral, and the induced map $M^* \to M^{gp}$ is injective if and only if M is quasi-integral. For any monoid M, the monoid $M/\sim$ (see 1.1.7) is sharp, and if M is integral, the natural map $M/M^* \to M/\sim$ is an isomorphism.

The inverse limit of a family of integral monoids is again integral. Note, however, that formation of M^{gp} does not commute with formation of inverse limits of integral monoids. For example, let $s \colon \mathbf{N}^2 \to \mathbf{N}$ be the map taking (a, b) to $a + b$ and let t be the map taking (a, b) to 0. Then the equalizer of s and t is the zero map. However, the equalizer of the associated maps on groups $\mathbf{Z}^2 \to \mathbf{Z}$ is the anti-diagonal $\mathbf{Z} \to \mathbf{Z}^2$ (sending c to $(c, -c)$.) On the other hand, it is true that an injective map $M \to N$ of integral monoids induces an injection $M^{gp} \to N^{gp}$.

Proposition 1.2.1 *If Q is an integral monoid and P is a submonoid, the natural map $Q/P \to Q^{gp}/P^{gp}$ is injective. Thus Q/P is integral and can be identified with the image of Q in Q^{gp}/P^{gp}. A monoid Q is integral if and only if it is quasi-integral and $\overline{Q}$ is integral.*

Proof: If q and q' are two elements of Q with the same image in Q^{gp}/P^{gp}, then there exist p and p' such that $q - q' = p - p'$ in Q^{gp}. Since Q is integral, $q + p' = q' + p$ in Q. Then it follows from (1.1.5) that q and q' have the same image in Q/P. In particular, if Q is integral, so is $\overline{Q}$. Conversely, suppose that Q is quasi-integral and $\overline{Q}$ is integral, and that q, q' and p are elements of Q with $q + p = q' + p$. Since $\overline{Q}$ is integral, there exists a unit u such that $q' = q + u$. Then $q' + p = q + p + u$. Since Q is quasi-integral, $u = 0$ and $q = q'$. This shows that Q is integral. □

Let $\mathbf{Mon}^{\mathrm{int}}$ denote the full subcategory of **Mon** whose objects are the integral monoids. For any monoid M, let M^{int} denote the image of $\lambda_M\colon M \to M^{gp}$. Then $M \mapsto M^{\mathrm{int}}$ is left adjoint to the inclusion functor $\mathbf{Mon}^{\mathrm{int}} \to \mathbf{Mon}$.

Proposition 1.2.2 *Let Q be the amalgamated sum of two homomorphisms $u_i\colon P \to Q_i$ in the category* **Mon**. *Then Q^{int} is the amalgamated sum of $u_i^{\mathrm{int}}\colon P^{\mathrm{int}} \to Q_i^{\mathrm{int}}$ in the category $\mathbf{Mon}^{\mathrm{int}}$, and can be naturally identified with the image of Q in $Q_1^{gp} \oplus_{P^{gp}} Q_2^{gp}$. If P, Q_1, and Q_2 are integral and any of these monoids is a group, then Q is integral.*

Proof: The fact that Q^{int} is the amalgamated sum of u_i^{int} in $\mathbf{Mon}^{\mathrm{int}}$ is a formal consequence of the fact that $M \mapsto M^{\mathrm{int}}$ preserves inductive limits. Moreover, since $M \mapsto M^{gp}$ also preserves inductive limits, $Q^{gp} \cong Q_1^{gp} \oplus_{P^{gp}} Q_2^{gp}$. It follows that Q^{int} is the image of Q in $Q^{gp} \cong Q_1^{gp} \oplus_{P^{gp}} Q_2^{gp}$. Now suppose that any of P and Q_i is a group and that (q_1, q_2) and (q_1', q_2') are two elements of $Q_1 \oplus Q_2$ with the same image in Q^{gp}. Then $v_1(q_1) + v_2(q_2) = v_1(q_1') + v_2(q_2')$ in Q^{gp}, and so there exist elements a and b in P such that $(q_1' - q_1, q_2' - q_2) = (u_1(a-b), u_2(b-a))$. Then $q_1' + u_1(b) = q_1 + u_2(a)$ and $q_2' + u_2(a) = q_2 + u_1(b)$. It then follows from (1.1.4) that $v_1(q_1) + v_2(q_2) = v_1(q_1') + v_2(q_2')$ in Q. Thus the map $Q \to Q_1^{gp} \oplus Q_2^{gp}$ is injective and Q is integral. □

A monoid M is said to be *fine* if it is finitely generated and integral. A monoid M is called *saturated* if it is integral and whenever $x \in M^{gp}$ is such that $mx \in M$ for some $m \in \mathbf{Z}^+$, then $x \in M$. For example, the monoid of all integers greater than or equal to some natural number d, together with zero, is not saturated if $d > 1$.

Proposition 1.2.3 *Let M be an integral monoid.*

1. *The natural map $M^{gp}/M^* \to \overline{M}^{gp}$ is an isomorphism.*
2. *If M is saturated, $\overline{M}^{gp}$ is torsion free.*
3. *The set M^{sat} of all elements x of M^{gp} such that there exists $n \in \mathbf{Z}^+$ with $nx \in M$ is a saturated submonoid of M^{gp}, and the functor $M \mapsto M^{\text{sat}}$ is left adjoint to the inclusion functor from the category $\mathbf{Mon}^{\text{sat}}$ of saturated monoids to $\mathbf{Mon}^{\text{int}}$.*
4. *M is saturated if and only if $\overline{M}$ is saturated.*
5. *The natural map $M^{\text{sat}}/M^* \to \overline{M}^{\text{sat}}$ is an isomorphism. Furthemore, every unit of $\overline{M}^{\text{sat}}$ is torsion, and the natural map*

$$\overline{M^{\text{sat}}} \to \overline{\overline{M}^{\text{sat}}}$$

is an isomorphism.

Proof: Suppose that $x_1, x_2 \in M$ and $x_2 - x_1$ maps to zero in $\overline{M}^{gp}$. Since $\overline{M} \subseteq \overline{M}^{gp}$, $\overline{x}_1 = \overline{x}_2 \in \overline{M}$, and hence there exists a $u \in M^*$ with $x_2 = u + x_1$. Then $x_2 - x_1 = u \in M^*$. This proves (1). Suppose M is saturated and $x \in M^{gp}$ maps to a torsion element of $\overline{M}^{gp}$. Then $nx \in M^*$ for some $n \in \mathbf{Z}^+$, and since M is saturated, $x \in M$. The fact that $nx \in M^*$ now implies that $x \in M^*$. Thus $\overline{M}^{gp}$ is torsion free. If x and y are elements of M^{gp} with $mx \in M$ and $ny \in M$, then $mn(x+y) \in M$, and it follows that M^{sat} is a submonoid of M^{gp}. Hence $(M^{\text{sat}})^{gp} = M^{gp}$, and if $x \in M^{\text{sat}}$ and $nx \in M^{\text{sat}}$, then there exists an $m \in \mathbf{Z}^+$ with $mnx \in M$. It follows that $x \in M^{\text{sat}}$, so M^{sat} is saturated. The verification of the adjointness of the functor $M \mapsto M^{\text{sat}}$ is immediate, as is that of (4). It is clear that $M^{\text{sat}}/M^* \to \overline{M}^{\text{sat}}$ is surjective, and the injectivity follows from the injectivity of the map $M^{gp}/M^* \to \overline{M}^{gp}$. Any element of $\overline{M}^{\text{sat}} \subseteq \overline{M}^{gp}$ is the image $\overline{x}$ of some element x of M^{gp}, and if $m\overline{x} \in \overline{M}$, $mx \in M$. If $\overline{x}$ is a unit of $\overline{M}^{\text{sat}}$, then there also exist an element y of M^{gp} and an $n \in \mathbf{Z}^+$ with $ny \in M$ and $x+y \in M^*$. Then $nmx+nmy \in M^*$

and hence $nmx \in M^*$. Since $mx \in M$, it follows that $mx \in M^*$ and that $m\overline{x} = 0$. Thus $\overline{x}$ is a torsion elemement of $\overline{M^{\mathrm{sat}}}$. It is clear that the map in (5) is surjective. Suppose that x and y are two elements of M^{sat} with the same image in $\overline{M^{\mathrm{sat}}}$. Then $x - y \in M^{gp}$ maps to a unit of $\overline{M^{\mathrm{sat}}}$, and hence to a torsion element of $\overline{M^{\mathrm{sat}}} \subseteq \overline{M^{gp}}$. Hence $mx - my \in M^*$ for some m. Then $my - mx \in M^*$ also, so $x - y$ is a unit of M^{sat}, and x and y have the same image in $\overline{M}^{\mathrm{sat}}$. The proves the injectivity. □

Monoids which are both fine and saturated are of central importance in logarithmic geometry, and are often called *normal* or *fs-monoids*. A monoid P is said to be *toric* if it is fine and saturated and in addition P^{gp} is torsion free; in this case P^{gp} can be viewed as the character group of an algebraic torus. The schemes arising from toric monoids form the building blocks of toric geometry.

A monoid M is said to be *valuative* if it is integral and for every $x \in M^{gp}$, either x or $-x$ lies in M. This is equivalent to saying that the order relation (1.1.7) on M^{gp} defined by the action of M is a total order. The monoid $\mathbf{N}$ is valuative, and if V is a valuation ring, the submonoid V' of nonzero elements of V is valuative. Every valuative monoid is saturated.

If R is any commutative ring, its underlying multiplicative monoid $(R, \cdot, 1)$ is not quasi-integral unless $R^* = \{1\}$, since $u \cdot 0 = 0$ for any $u \in R^*$, and it is not integral unless $R = \{0\}$, since $0 \cdot 0 = 1 \cdot 0$. On the other hand, the set R' of nonzero divisors of R forms an integral submonoid of the multiplicative monoid of R. For example, $\overline{\mathbf{Z}}' = \mathbf{Z}'/(\pm)$ is a free (commutative) monoid, generated by the prime numbers. If R is a discrete valuation ring, $\overline{R}' = R'/R^*$ is freely generated by the image of a uniformizer of R'. Although there is a unique isomorphism of monoids $R'/R^* \cong \mathbf{N}$, it is not functorial: if $R \to S$ is a finite extension of valuation rings with ramification index e, the induced map $\overline{R}' \to \overline{S}'$ sends the unique generator of $\overline{R}'$ to e times that of $\overline{S}'$.

1.3 Ideals, faces, and localization

Definition 1.3.1 *An ideal of a monoid M is a subset I such that $x \in I$ and $y \in M$ implies $x + y \in I$. An ideal I is called prime if $I \neq M$ and $x + y \in I$ implies $x \in I$ or $y \in I$. A face of a monoid M is a submonoid F such that $x + y \in F$ implies that both x and y belong to F.*

Observe that a face is just a submonoid whose complement is an ideal, and a prime ideal is an ideal whose complement is a submonoid (hence a face). Thus $\mathfrak{p} \mapsto F_{\mathfrak{p}} := M \setminus \mathfrak{p}$ gives an order reversing bijection between the set of prime ideals of M and the set of faces of M. The empty set is an ideal of M—the unique minimal prime ideal. The set of units M^* is a face of M, and in fact is contained in every face. Its complement, the set M^+ of all nonunits of M, is an ideal of M, and in fact contains every proper ideal of M. Thus M^+ is the unique maximal ideal of M; in many respects a monoid is analogous to a local ring. The notion of a face of a monoid corresponds to the notion of a saturated multiplicative subset of a ring; we do not use this terminology here because of its conflict with the notion of a saturated monoid defined above.

The union of a family of prime ideals is a prime ideal and the intersection of a family of faces is a face. The intersection of all the faces containing some subset T of M is a face, called the *face generated by* T; it is analogous to the multiplicatively saturated set generated by a subset of a ring. The *interior* I_M of a monoid M is the set of all elements which do not lie in a proper face of M, *i.e.*, the intersection of all the nonempty prime ideals of M.

We denote by $\operatorname{Spec}(M)$ the set of prime ideals of a monoid. If I is an ideal of M and $Z(I)$ denotes the set of primes of M containing I, one finds in the usual way that the set of subsets $Z(I)$ of $\operatorname{Spec}(M)$ defines a topology on $\operatorname{Spec}(M)$ (the Zariski topology), in which the irreducible closed sets correspond to the prime ideals. Since M has a unique minimal prime ideal, $\operatorname{Spec}(M)$ has a unique generic point, and in particular is irreducible. If $\theta\colon M \to N$ is a morphism of monoids, then the inverse image of an ideal is an ideal, the inverse image of a prime ideal is a prime ideal, and the inverse image of a face is a face. Thus θ induces a continuous map

$$\operatorname{Spec} N \to \operatorname{Spec} M : \quad \mathfrak{p} \mapsto \theta^{-1}(\mathfrak{p}).$$

If $\theta\colon M \to N$ is a morphism of monoids, then $\theta^{-1}(N^*)$ is a face of M, and $\theta\colon M \to N$ is said to be *local* if $\theta^{-1}(N^*) = M^*$, or, equivalently, if $\theta^{-1}(N^+) = M^+$.

The order relation 1.1.7 is useful when describing ideals and faces of a monoid.

Proposition 1.3.2 *Let S be a subset of a monoid Q and let P be the submonoid of Q generated by S.*

1. *The ideal (S) of Q generated by S is the set of all $q \in Q$ such that $q \geq s$ for some $s \in S$.*

2. *The face $\langle S\rangle$ of Q generated by S is the set of elements q of Q for which there exists a $p \in P$ such that $q \leq p$. In particular, the face generated by an element p of Q is the set of all elements $q \in Q$ such that $q \leq np$ for some $n \in \mathbf{N}$.*

3. *If Q is integral, then Q/P is sharp if and only if $P^{gp} \cap Q$ is a face of Q. In particular, if F is a face of Q, then Q/F is sharp.*

Proof: The first statement follows immediately from the definitions. For the second, note that a submonoid F of Q is a face if and only if $q \leq f$ with $f \in F$ implies that $q \in F$. Hence $\langle S\rangle$ contains the set P' of all $q \in Q$ such that there exists a $p \in P$ with $q \leq p$. Since in fact P' is necessarily a submonoid of Q, it is also a face, so $P' = \langle S\rangle$. If Q is integral, Q/P can be identified with the image of Q in Q^{gp}/P^{gp}, by 1.2.1. Thus an element $q \in Q$ maps to a unit in Q/P if and only if there exists an element $q' \in Q$ such that $q + q' \in P^{gp}$, *i.e.*, if and only if $q \leq q''$ for some $q'' \in Q \cap P^{gp}$. This shows that Q/P is sharp if and only if $Q \cap P^{gp}$ is a face of Q. Finally, note that if F is a face of Q, and $q \in Q \cap F^{gp}$, then $q + f \in F$ for some $f \in F$, hence $q \in F$. □

Proposition 1.3.3 *Let M be a monoid, S a subset of M, and E an M-set. Then there exists an M-set $S^{-1}E$ on which the elements of S act bijectively and a map of M-sets $\lambda_S \colon E \to S^{-1}E$ which is universal: for any morphism of M-sets $E \to E'$ such that each $s \in S$ acts bijectively on E', there is a unique M-map $S^{-1}E \to E'$ such that*

$$\begin{tikzcd}
E \arrow[r, "\lambda_S"] \arrow[dr] & S^{-1}E \arrow[d] \\
 & E'
\end{tikzcd}$$

commutes. The morphism λ_S is called the localization of E by S.

Proof: Let T be the submonoid of M generated by S. The set $S^{-1}E$ can be constructed in the familiar way as the set of equivalence classes of pairs

$(e,t) \in E \times T$, where $(e,t) \equiv (e',t')$ if and only if $\theta(t't'')e = \theta(tt'')e'$ for some t'' in T. Then $\lambda_S(e)$ is the class of $(e,0)$, and the action of an element m of M sends the class of (e,t) to the class of $(\theta(m)e,t)$. $\square$

Notice that in fact every element of the face F generated by S acts bijectively on $S^{-1}E$, so that in fact $S^{-1}E \cong F^{-1}E$. Indeed, if $f \in F$, then $f \leq t$ for some t in the submonoid T of M generated by S. Thus $t = fm$ for some $m \in M$. Then $\theta_S(t) = \theta_S(f)\theta_S(m) = \theta_S(m)\theta_S(f)$, and since $\theta_S(t)$ is bijective, the same is true of $\theta_S(f)$. If $\mathfrak{p} := M \setminus F$ is the prime ideal of M corresponding to F, one often writes $E_{\mathfrak{p}}$ instead of $S^{-1}E$. An M-set E is called *M-integral* if the elements of M act as injections on E. If this is the case, the localization map $\lambda_S \colon E \to S^{-1}E$ is injective, for every subset S of M.

The most important case of (1.3.3) is the case where E is M itself with the action of M on itself by translations. Then $S^{-1}M$ has a unique monoid structure for which λ_S is a morphism. If $\theta \colon M \to N$ is a morphism of monoids and S is a subset of M we write $S^{-1}N$ to mean the localization of N by the image of S, when no confusion can arise.

Remark 1.3.4 The localization of an integral (resp. saturated) monoid is integral (resp. saturated), but the analog for quasi-integral monoids fails, as the following example shows.

Let Q and P be monoids and let K be an ideal of Q. Let E be the subset of $(P \oplus Q)$ consisting of those pairs $(p \oplus q, p' \oplus q)$ such that either $p = p'$ or $q \in K$. In fact E is a congruence relation on $P \oplus Q$, and we denote the quotient $(P \oplus Q)/E$ by $P \star_K Q$ (the *join of P and Q along K*). If K is a prime ideal with complement F, then $P \star_K Q$ can be identified with the disjoint union of $P \times F$ with K, and $(p,f) + k = f + k$. Then $\mathbf{N} \star_{\mathbf{N}^+} \mathbf{N}$ is quasi-integral, but its localization by the element 1 of the "first" $\mathbf{N}$ is $\mathbf{Z} \star_{\mathbf{N}^+} \mathbf{N}$, which is not quasi-integral.

Definition 1.3.5 *Let M be a monoid.*

1. *The dimension of M is the Krull dimension of the topological space* $\operatorname{Spec}(M)$, *i.e., the maximum length h of a chain of prime ideals* $\emptyset = \mathfrak{p}_0 \subset \mathfrak{p}_1 \cdots \subset \mathfrak{p}_d = M^+$.

2. *If* $\mathfrak{p} \in \operatorname{Spec}(M)$, $\operatorname{ht}(\mathfrak{p})$ *is the maximum length of a chain of prime ideals* $\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \mathfrak{p}_h$.

If $\mathfrak{p}$ is a prime ideal of M, the map $\mathrm{Spec}(M_\mathfrak{p}) \to \mathrm{Spec}(M)$ induced by the localization map $\lambda\colon M \to M_\mathfrak{p}$ is injective and identifies $\mathrm{Spec}(M_\mathfrak{p})$ with the subset of $\mathrm{Spec}(M)$ consisting of those primes contained in $\mathfrak{p}$. Equivalently, $F \mapsto \lambda^{-1}(F)$ is a bijection from the set of faces of $M_\mathfrak{p}$ to the set of faces of M containing $M \setminus \mathfrak{p}$. These bijections are order preserving. In particular, we have $\mathrm{ht}(\mathfrak{p}) = \dim(M_\mathfrak{p})$. If M is fine, $\mathrm{Spec}\, M$ is a finite topological space, and is catenary (even biequidimensional, in the sense of [5, 14.3.2, 14.3.3]), as the following proposition shows. We defer its proof until section (2.3).

Proposition 1.3.6 *Let M be a fine monoid.*

1. $\mathrm{Spec}\, M$ *is a finite set.*
2. $\dim(M) = \mathrm{rank}\overline{M}^{gp}$, *where* $\overline{M}^{gp} \cong M^{gp}/M^*$.
3. *Any prime* $\mathfrak{p} \in \mathrm{Spec}\, M$ *is contained in a chain* $\mathfrak{p}_0 \subset \mathfrak{p}_1 \cdots \subset \mathfrak{p}_d$ *of length* $\dim(M)$.
4. *For any* $\mathfrak{p} \in \mathrm{Spec}\, M$,

$$\mathrm{ht}(\mathfrak{p}) + \dim(F_\mathfrak{p}) = \dim(M).$$

Examples 1.3.7 The monoid $\mathbf{N}$ has just two faces, $\{0\}$ and $\mathbf{N}$. More generally, let S be a finite set and let $M := \mathbf{N}^{(S)}$, the free monoid generated by S. If T is any subset of S, $\mathbf{N}^{(T)}$ can be identified with the set of all $I \in \mathbf{N}^{(T)}$ such that $I_s = 0$ for $s \notin T$. This is a face of M, and every face of M is of this form. A more complicated example is provided by the monoid P which is given by generators x, y, z, w subject to the relation $x+y = z+w$. This monoid is isomorphic to the submonoid of $\mathbf{N}^4$ generated by $\{(1,1,0,0),(0,0,1,1),(1,0,1,0),(0,1,0,1)\}$ and to the submonoid of $\mathbf{Z}^3$ generated by $\{(1,1,1),(-1,-1,1),(1,-1,1),(-1,1,1)\}$. It is also the coproduct $\mathbf{N}^2 \oplus_\mathbf{N} \mathbf{N}^2$, where both maps $\mathbf{N} \to \mathbf{N}^2$ send 1 to $(1,1)$. In addition to the faces $\{0\}$ and P, it has four faces of dimension one, corresponding to each of the generators, and four faces of dimension two: $\langle x, z\rangle, \langle x, w\rangle, \langle y, z\rangle, \langle y, w\rangle$. For yet another example, consider the monoid Q given by generators x, y, z, u, v subject to the relations $x + y + z = u + v$. This four-dimensional monoid has five faces of dimension one and nine of dimensions two and three.

2 Convexity, finiteness, and duality

2.1 Finiteness

Proposition 2.1.1 *A quasi-integral monoid is finitely generated as a monoid if and only if M^* is finitely generated (as a group) and $\overline{M}$ is finitely generated (as a monoid).*

Proof: If M is finitely generated as a monoid, then M^{gp} is finitely generated as a group. Since M is quasi-integral, $M^* \subseteq M^{gp}$, and it follows that M^* is finitely generated as a group. Since $M \to \overline{M}$ is surjective, $\overline{M}$ is finitely generated as a monoid. For the converse, suppose $\{s_i\}$ is a finite set of generators for the group M^* and $\{t_j\}$ is a finite subset of M whose images in $\overline{M}$ generate $\overline{M}$ as a monoid. Then $\{s_i, -s_i, t_j\}$ generates M as a monoid. $\square$

Recall that if x and y are two elements of a monoid M, we write $x \leq y$ if there exists a $z \in M$ such that $y = x + z$. If S is a subset of a monoid M and $s \in S$, we say that s is a *minimal element of S* (or *M-minimal* if we need to specify the monoid) if whenever $s' \in S$ and $s' \leq s$, then also $s \leq s'$ (so that $s \sim s'$ in the equivalence relation corresponding to $\leq$).

An M-minimal element of the maximal ideal M^+ of an integral monoid M is called an *irreducible element* of M. An element c of M is irreducible if and only if it is not a unit and whenever $c = a + b$ in M, a or b is a unit.

Proposition 2.1.2 *Let M be a sharp integral monoid. Then every set of generators of M contains every irreducible element of M. If in addition M is finitely generated, then the set of irreducible elements of M is finite and generates M.*

Proof: The first statement is obvious. Suppose now that M is finitely generated. It is clear that every finite set of generators contains a minimal set of generators. Let S be such a minimal set; we claim that every element x of S is irreducible. If $x = y + z$ with y and z in M, we can write $y = \sum_s a_s s$ and $z = \sum_s b_s s$, where a_s and $b_s \in \mathbf{N}$ for all $s \in S$. Then $x = \sum_s c_s s$, where $c_s = a_s + b_s$. Let $S' := S \setminus \{x\}$, so that $(1 - c_x)x = \sum\{c_s s : s \in S'\}$ in M^{gp}. If $c_x > 1$ we see that x is a unit, and since M is sharp, $x = 0$ and S' generates M, a contradiction. If $c_x = 0$, $x = \sum\{c_s s : s \in S'\}$, again contradicting the minimality of S. It follows that $c_x = 1$, and hence we can write $y = y' + a_x x$

and $z = z' + b_x x$, where $a_x + b_x = 1$. Then $x = y + z = x + y' + z'$, so $y' + z' = 0$, and since M is sharp, $y' = z' = 0$. This shows that y or z is zero, so x is irreducible, as claimed. Since S contains all the irreducible elements of M, there can be only finitely many such elements. □

Corollary 2.1.3 *The automorphism group of a fine sharp monoid is finite, contained in the permutation group of the set of its irreducible elements.*

Remark 2.1.4 Proposition (2.1.2) shows that every element in a fine sharp monoid can be written as a sum of irreducible elements. In fact a standard argument applies somewhat more generally. Let M be a sharp integral monoid in which every nonempty subset contains a minimal element. Then every element of M can be written as a sum of irreducible elements. (Note that 0 is by definition the sum over the empty set of irreducible elements.) Let us recall the argument. We claim that the set S of elements of M^+ which cannot be written as a sum of irreducible elements is empty. If not, by assumption it contains a minimal element s. Since s is not irreducible, $s = a + b$ where a and b are not zero. If both a and b can be written as sums of irreducible elements, then the same is true of s, a contradiction. But if say a cannot be written as a sum of irreducible elements, $a \in S$ and $a \leq s$ and s is not less than or equal to a, a contradiction of the minimality of s.

Proposition 2.1.5 *Let M be a finitely generated monoid.*

1. *Any sequence $(s(1), s(2), \ldots)$ of elements of M contains an increasing subsequence $(s(i_1) \leq s(i_2) \leq s(i_3) \leq \cdots)$.*
2. *Any decreasing sequence $s(1) \geq s(2) \geq s(3), \ldots$ in M lies eventually in a single equivalence class for the relation $\sim$.*
3. *Any nonempty subset S of M contains a minimal element, and there are only finitely many equivalence classes (for the relation $\sim$) of such elements.*
4. *If M is integral and sharp, any decreasing sequence in M is eventually constant, and any nonempty subset of M has a finite nonzero number of minimal elements.*

Proof: We begin by proving (2.1.5.1), which was pointed out to us by H. Lenstra, when $M = \mathbf{N}^r$. Let $s_1 := pr_1 \circ s$ be the sequence of first coordinates of s. Let n_1 denote the minimum of the set of all $s_1(i)$ for $i \in \mathbf{Z}^+$, and choose i_1 with $s_1(i_1) = n_1$. Let n_2 be the minimum of the set of all $s_1(i)$ with $i > i_1$, and choose $i_2 > i_1$ with $s_1(i_2) = n_2$. Continuing in this way, we find a sequence $1 \leq i_1 < i_2 \cdots$ such that $s_1(i_1) \leq s_1(i_2) \leq s_1(i_3) \cdots$. Replacing s by its subsequence $s(i_1), s(i_2), \ldots$, we may assume that s has the property that s_1 is increasing. Now repeat this process with the sequence of second coordinates, and we find that both s_1 and s_2 are increasing. After doing this with each i in succession, we find that s_i is increasing for every i, and hence that s is increasing. If M is any finitely generated monoid, there is a surjective morphism $\theta\colon \mathbf{N}^r \to M$, and any sequence s in M can be lifted to a sequence t in $\mathbf{N}^r$. We have just seen that t has an increasing subsequence t',, and the image of t' in M is an increasing subsequence of s.

The remaining statement are formal consequences of the first. To prove (2), let $s(1), s(2), \ldots$ be a sequence of elements in M such that $s(1) \geq s(2) \geq s(3) \cdots$. It suffices to show that there exists $i \in \mathbf{Z}^+$ such that $s(j+1) \geq s(j)$ for all $i \geq j$. If not, then there exists a sequence $i_1 < i_2 < \cdots$ such that $s(i_k + 1) \not\geq s(i_k)$ for all k. By (1), the sequence $s(i_1), s(i_2), \cdots$ has an increasing subsequence, and hence there certainly exist k and k' such that $k < k'$ and $s(i_k) \geq s(i_k)$. But then $i_k + 1 \leq i_{k'}$, so $s(i_k + 1) \geq s(i_{k'}) \geq s(i_k)$, a contradiction.

If S is a nonempty subset of M, choose any element $s(1)$ of S. If $s(1)$ is M-minimal, we are done; if not there exists an element $s(2)$ of S such that $s(2) \leq s(1)$ and $s(2) \not\geq s(1)$. If $s(2)$ is M-minimal, we are done, and if not there exists $s(3)$ with $s(3) \leq s(2)$ and $s(3) \not\geq s(2)$. Continuing in this way, we find a decreasing sequence $s(1) \ldots s(n)$ of elements of S with $s(i) \not\geq s(i-1)$ for $i = 1, \ldots n$. By (2), such a sequence must terminate, and then $s(n)$ is an M-minimal element of S. If there were an infinite number of equivalence classes of such minimal elements, we could find an infinite sequence s of elements all belonging to distinct equivalence classes, and by (1) such a sequence would contain an increasing subsequence s. But then $s(1) \leq s(2)$ and $s(1) \not\sim s(2)$, contradicting the minimality of $s(2)$. This proves (3), and (4) follows. □

Remark 2.1.6 An action of a monoid Q on a set S defines a preorder $\leq$ on S: $s \leq t$ if there exists $q \in Q$ such that $q + s = t$. If we let Q act on itself via the regular representation, this definition is the same as the order relation used for monoids, and if $h\colon S \to T$ is a morphism of Q sets, then $s \le s'$ implies $h(s) \le h(s')$, and conversely if h is injective. Furthermore, statements (1), (2), and (3) make sense and are valid for any finitely generated Q-set S. To see this, use the fact that if S is finitely generated as a Q-set, then there exists $r \in \mathbf{N}$ and a surjective map of Q-sets $f\colon \cup_r Q \to S$, where $\cup_r Q$ is the disjoint union of r copies of Q acting regularly on itself. A sequence of elements of S admits a subsequence which lies in the image of one of the copies of Q. Thus (1) for S follows from (1) for Q, and (2) and (3) are formal consequences.

Remark 2.1.7 Let S be a nonempty subset of a monoid M, and suppose that M is a submonoid of a sharp fine monoid N. Since N is fine, Proposition 2.1.5 shows that S contains an N-minimal element, and such an element is also necessarily M-minimal. (If $s = m + s'$ with $m \in M$ and $s' \in S$, then there exist $n \in N$ such that $s' = n + s$, hence $m + n = 0$ and $m = n = 0$.) In particular, Remark 2.1.4 implies that M is generated by its irreducible elements. On the other hand, M-minimal elements of S need not be N-minimal, and it could happen that S has an infinite number of minimal elements and that M has an infinite number of irreducible elements. For example, in $N := \mathbf{N} \times \mathbf{N}$, consider the submonoid M of $\mathbf{N} \times \mathbf{N}$ consisting of $(0,0)$ together with all pairs (m,n) such that m and n are both positive. (This submonoid is even a congruence relation on $\mathbf{N}$; the quotient $\mathbf{N}/M$ is the unique (up to isomorphism) monoid with two elements which is not a group.) Then for every $m > 0$, the element $(1,m)$ is irreducible in M, and in particular M is not finitely generated as a monoid. This situation is illuminated by the notion of *exactness*, which will turn out to be of fundamental importance in logarithmic geometry.

Definition 2.1.8 *A morphism of monoids $f\colon M \to N$ is exact if the diagram*

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\
M^{gp} \arrow[r, "f^{gp}"] & N^{gp}
\end{tikzcd}$$

is Cartesian.

Note that the diagonal morphism $\Delta_M \colon M \to M \times M$ is exact if and only if the map $M \to M^{gp}$ is injective, *i.e.*, if and only if M is integral. If M and N are integral, then f is exact if and only if whenever x and y are elements of M, $f(x) \leq f(y)$ implies that $x \leq y$. If M is a submonoid of an integral monoid N, then $M \to N$ is exact if and only if $M = M^{gp} \cap N$. Note also that the canonical morphism $M \to \overline{M}$ is exact.

Theorem 2.1.9

1. *Every ideal in a finitely generated monoid is finitely generated (as an ideal).*
2. *Every exact submonoid of a fine (resp. saturated) monoid is finitely generated (resp. saturated).*
3. *A face of an integral monoid is an exact submonoid. Every face of a fine monoid is finitely generated (as a monoid), and monogenic (as a face).*
4. *Every localization (1.3.3) of a fine monoid (resp. saturated) is fine (resp. saturated).*
5. *The equalizer of two maps from a fine (resp. saturated) monoid to an integral monoid is fine (resp. saturated).*
6. *The fiber product of two fine (resp. saturated) monoids over an integral monoid is fine (resp. saturated).*
7. *Any congruence relation on a finitely generated monoid is finitely generated (as a congruence relation). In particular, any finitely generated monoid is finitely presented.*
8. *Let P and Q be monoids. If Q is fine and P is finitely generated,* $\mathrm{Hom}(P, Q)$ *is also fine. If Q is saturated,* $\mathrm{Hom}(P, Q)$ *is also saturated.*

Proof: First observe that any ideal I of a finitely generated monoid M is generated by the set S of its minimal elements. Indeed, if I' is the ideal of M generated by S, then $I' \subseteq I$, and if $I \setminus I'$ is not empty, (2.1.5.3) implies that it contains a minimal element t. Since t does not belong to S, it is not minimal as an element of I, so there exists some $q \in I$ such that $q \leq t$

and $t \not\leq q$. The minimiality of t in $I \setminus I'$ implies that $q \notin I \setminus I'$. But then $q \in I'$ and consequently also $t \in I'$, which is a contradiction. Notice that two elements s and s' of S with $s \sim s'$ generate the same ideal. Thus a subset T of S containing one element from each equivalence class will still generate I and will be finite by (2.1.5.3).

Next we observe that if S is a subset of an exact submonoid M of a fine sharp monoid N, the set of minimal elements of S is finite. In fact, if x and y are two elements of M and $x \leq y$ in N then also $x \leq y$ in M. Thus any M-minimal element of S is also N-minimal, and by (2.1.5) the set of these is finite. In particular, the set of irreducible elements of M is finite, and by Proposition (2.1.2) it follows that M is finitely generated. This proves that every exact submonoid of a fine sharp monoid is finitely generated. Slightly more generally, if M is an exact submonoid of any fine monoid N, we can choose a surjection $\mathbf{N}^r \to N$, and the inverse image M' of M in $\mathbf{N}^r$ is an exact submonoid of $\mathbf{N}^r$. It follows that M' is finitely generated, and hence so is M. Suppose now that N is saturated and $x \in M^{gp}$ with $nx \in M$ for some $n \in \mathbf{Z}^+$. Then $x \in N \cap M^{gp} = M$, so M is also saturated. This proves (2).

If F is a face of an integral monoid M and if x and y belong to F and $z := x - y \in M$, then $x = y + z \in F$. Since F is a face, it follows that $z \in F$, so F is an exact submonoid of M. Hence F is finitely generated as a monoid. If $f_1, \ldots f_n$, are generators, then $f := f_1 + \cdots + f_n$ generates F as a face of M. If $S \subseteq M$ is a finite set of generators of M, then $F^{-1}M$ is generated by the set of elements of the form $\lambda(s) - \lambda(f)$ with $s \in S$. This proves the third and fourth statements, since localization preservations saturation.

Let $E \to P$ be the equalizer of two maps θ_1 and θ_2 from P to M, with M integral and P finitely generated. Then $E \to P$ is just the pullback of the diagonal Δ_M via the map $(\theta_1, \theta_2)\colon P \to M \times M$, and since Δ_M is exact, so is $E \to P$. This proves the fifth statement, since an exact submonoid of a fine (resp saturated) monoid is fine (resp saturated). The sixth follows because the product of two fine monoids is fine.

The following short proof of (7) is due to Pierre Grillet [3]. We may assume without loss of generality that P is finitely generated and free, hence isomorphic to $\mathbf{N}^r$. If p and q are elements of P, write $p \preceq q$ if p precedes q in the lexicographical order of $\mathbf{N}^r$, and write $p \prec q$ if in addition $p \neq q$. If $p \preceq q$ and $p' \preceq q'$, then $p + p' \preceq q + q'$, and if $p \leq q$ in the partial order defined by the monoid structure, then $p \preceq q$. The order relation $\preceq$ is a well-orders $\mathbf{N}^r$: every nonempty subset has a unique $\preceq$-minimal element. If E is a congruence relation on P and $p \in P$, let $E(p)$ denote the E-congruence class of p, and let $\mu(p)$ denote the $\preceq$-minimial element in $E(p)$. The complement K of the image of $\mu\colon P \to P$ is the set of all elements k of P such that $\mu(k) \prec k$. Note that if $p \in P$ and $\mu(k) \prec k$, then $\mu(k) + p \prec k + p$, and since $(\mu(k) + p) \equiv_E (k + p)$, $k + p$ is not $\preceq$-minimal in $E(k + p)$. Thus $\mu(k + p) \prec k + p$ and so K is an ideal of P. The congruence relation E' on P generated by the set of pairs $(s, \mu(s))$ with s taken from a finite set S of generators for K is finitely generated and contained in E, so it will suffice to prove that $E \subseteq E'$, *i.e.*, that E' contains $(x, \mu(x))$ for every $x \in P$. If this fails, there exists an x such that $\mu(x) \notin E'(x)$ and which is $\preceq$-minimal among all such elements. Evidently x does not belong the image of μ, so $x \in K$, and hence $x = p + s$ for some $s \in S$ and $p \in P^+$. Since $\mu(s) \prec s$, $x' := p + \mu(s) \prec p + s = x$, and hence by the minimality of x, $E'(x')$ contains $\mu(x')$. But $\mu(s) \equiv_{E'} s$, so $x' \equiv_{E'} x$, and it follows that $\mu(x') = \mu(x)$ and that $\mu(x) \in E'(x)$, a contradiction.

It is clear that $\mathrm{Hom}(P, Q)$ is integral (resp. saturated) if Q is integral (resp. saturated). If P is finitely generated, choose a surjective map $\mathbf{N}^r \to P$ for some $r \in \mathbf{Z}^+$. Then $\mathrm{Hom}(P, Q)$ can be identified with the equalizer of the two maps $\mathrm{Hom}(\mathbf{N}^r, Q) \to \mathrm{Hom}(\mathbf{N}^r \times_P \mathbf{N}^r, Q)$. Since $\mathrm{Hom}(\mathbf{N}^r, Q) \cong Q^r$ is finitely generated if Q is, the same is true of $\mathrm{Hom}(P, Q)$, by (6). $\square$

Remark 2.1.10 If Q is a finitely generated monoid and S is a finitely generated Q-set, then any invariant Q-subset of S is finitely generated as a Q-set. This can be proved in the same way as (2.1.9.1), using (2.1.6).

Remark 2.1.11 If P is an integral monoid and E is a congruence relation on P, then P/E is integral if and only if $E \to P \times P$ is exact. Indeed the congruence relation E determined by a surjective map $\theta\colon P \to Q$ of integral monoids is just the equalizer of the two maps $P \times P \to Q$, and we saw in the proof of (2.1.9.5) that it is then an exact submonoid of $P \times P$. For the converse, suppose that $E \to P \times P$ is exact and $\theta\colon P \to Q$ is the coequalizer of the two maps $E \to P$. If $\theta(p_1) + \theta(p) = \theta(p_2) + \theta(p)$ in Q, then $e := (p_1, p_2) + (p, p) \in E$. Since $(p, p) \in E$, it follows that $(p_1, p_2) \in E^{gp} \cap P \times P$, and hence that $(p_1, p_2) \in E$. Then $\theta(p_1) = \theta(p_2)$, so Q is integral. In particular congruence relations on P yielding integral quotients Q correspond to congruence relations on P^{gp} and hence by (1.1.3) to subgroups of P^{gp}. Of course, the subgroup of P^{gp} corresponding to a surjective map of integral monoids $P \to Q$ is just the kernel of $P^{gp} \to Q^{gp}$.

Corollary 2.1.12 *Let P be a fine monoid and let E be a congruence relation on P such that P/E is integral. E is finitely generated as a monoid (not just as a congruence relation).*

Corollary 2.1.13 *Let $P \to M$ be a morphism of integral monoids. If P and $\overline{M}$ are finitely generated, then so is $P^{gp} \times_{M^{gp}} M$.*

Proof: It suffices to observe that the map $P^{gp} \times_{M^{gp}} M \to P^{gp} \times_{\overline{M}^{gp}} \overline{M}$ is an isomorphism and to apply (2.1.9.4) and (2.1.9.6). ☐

Proposition 2.1.14 *Let Q be a sharp valuative monoid. Then the following conditions are equivalent.*

- *Q is isomorphic to* $\mathbf{N}$.
- *Q^{gp} is isomorphic to* $\mathbf{Z}$.
- *Q is finitely generated.*

Proof: It is evident that (1) implies (2). If (2) holds, let $\nu\colon \mathbf{Q}^{gp} \to \mathbf{Z}$ be an isomorphism and choose $q \in Q^{gp}$ with $\nu(q) = 1$. Either q or $-q$ lies in Q, so by changing the signs of q and/or ν we may arrange things so that $q \in Q$ and $\nu(q) = 1$. Then the sharpness of Q implies that $\nu(q') \geq 0$ for all $q' \in Q$. Thus ν induces a homomorphism $Q \to \mathbf{N}$ which is necessarily bijective. This proves the equivalence of (1) and (2). Suppose that (3) holds. Since Q is valuative, the order relation on Q is a total order, and Proposition (2.1.5.3) implies that it is even a well-ordering. Thus Q^+ has a unique minimal element which then (freely) generates Q. This proves the equivalence of (1) and (3). ☐

Example 2.1.15 Let X be a normal locally noetherian scheme and Y a proper closed subset. Then it follows from Theorem (2.1.9.2) that the stalks of the sheaf $\underline{\Gamma}_Y Div_X^+$ of effective Cartier divisors on X with support in Y are fine monoids. To see this, let $\mathcal{O}'_X$ be the subsheaf of $\mathcal{O}_X$ which to each open set U of X assigns the set of sections f such that $f_x \neq 0 \in \mathcal{O}_{X,x}$ for all $x \in U$. This a sheaf of submonoids of $\mathcal{O}_X$, and Div_X^+ can be identified with the quotient $\mathcal{O}'_X/\mathcal{O}^*_X$. Let $\mathcal{W}^+_X$ be the sheaf of effective Weil divisors, *i.e.*, the sheaf associated to the presheaf which to every open U assigns the free monoid on the set of points $\eta \in U$ such that $\mathcal{O}_{U,\eta}$ has dimension one. Since X is regular in codimension one, each $\mathcal{O}_{U,\eta}$ is a discrete valuation ring, and the valuation maps induce a morphism of monoids $\nu\colon \mathcal{O}'_X \to \mathcal{W}^+_X$ [7, II §6]. The normality of X implies that for any $x \in X$, $\mathcal{O}_{X,x}$ is the intersection, in the fraction field $K_{X,x}$ of $\mathcal{O}_{X,x}$, of its localizations at height one primes. It follows that $\mathcal{O}'_{X,x}$ is the set of sections f of $K_{X,x}$ such that $\nu^{gp}(v) \in \mathcal{W}^+_X$, and that $\mathcal{O}^*_X$ is the kernel of ν. Hence the morphism $\nu_x\colon \mathcal{O}'_{X,x} \to \mathcal{W}^+_{X,x}$ is exact, and $Div^+ := \mathcal{O}'_{X,x}/\mathcal{O}^*_{X,x}$ is an exact submonoid of $\mathcal{W}^+_{X,x}$, and hence the stalk at x $\underline{\Gamma}_Y(Div^+_X)$ is an exact submonoid of the stalk at x of $\underline{\Gamma}_Y(\mathcal{W}^+_X)$. The latter is just the free monoid on the set of prime ideals of height one in the local ring $\mathcal{O}_{X,x}$ which are contained in Y. Since Y is a proper closed subset of X, each of these is a minimal prime of the noetherian local ring $\mathcal{O}_{Y,x}$, and hence there only finitely many such primes. Thus $\underline{\Gamma}_Y(\mathcal{W}^+_X)_x$ is a fine monoid, and by (2.1.9.2), the same is true of $\underline{\Gamma}_Y(Div^+_X)_x$.

To see that the normality hypothesis is not superfluous, let X be the spectrum of the subring R of $\mathbf{C}[t]$ consisting of those polynomials whose first derivative vanishes at $t = 0$. This is a curve with a cusp at the origin x. Let $Y := \{x\}$ and for any complex number a, let D_a be the class of $t^2 - at^3$ in $Div^+_{X,x} = \mathcal{O}'_{X,x}/\mathcal{O}^*_{X,x}$. Note that in $K_{X,x}$, $(t^2 - at^3)/(t^2 - bt^3) = (1 - at)/(1 - bt) = 1 + (b - a)t + \cdots$, which does not belong to $\mathcal{O}^*_{X,x}$ if $a \neq b$. Thus $D_a \neq D_b \in \underline{\Gamma}_Y(Div^+_X)_x$. It follows that $\underline{\Gamma}_Y(Div^+_X)_x$ is uncountable and hence is not finitely generated. Similar examples can be made with local nodal curves.

2.2 Duality

Duality, and in particular the existence of "enough" homomorphisms from a fine monoid to $\mathbf{N}$, is a crucial tool in the theory of toric varieties.

Theorem 2.2.1 *Let Q be a fine monoid, and let $H(Q) := \operatorname{Hom}(Q, \mathbf{N})$.*

1. *The monoid $H(Q)$ is fine, saturated, and sharp.*
2. *The natural map $H(Q)^{gp} \to \operatorname{Hom}(\overline{Q}^{gp}, \mathbf{Z})$ is an isomorphism.*

3. *The evaluation mapping* $ev\colon Q \to H(H(Q))$ *factors through an isomorphism*
$$\overline{ev}\colon \overline{Q^{\mathrm{sat}}} \to H(H(Q)).$$

Proof: The key geometric tool is the following. Let P be a submonoid of an abelian group G, and let ϕ be a homomorphism $G \to \mathbf{Z}$ which maps P to $\mathbf{N}$. Suppose that t is an element of G and $\phi(t) < 0$, and let Q be the submonoid of G generated by P and t. Then the homomorphism

$$\psi\colon G \to \mathrm{Ker}(\phi) : g \mapsto t\phi(g) - g\phi(t)$$

induces multiplication by $|\phi(t)|$ on $\mathrm{Ker}(\phi)$ and maps Q into P.

The following result is a corollary of the theorem, but in fact it is one of the main ingredients in the proof.

Lemma 2.2.2 *If Q is a fine monoid, there exists a local homomorphism $h\colon Q \to \mathbf{N}$; i.e., an element of $H(Q)$ such that $h^{-1}(0) = Q^*$.*

Proof: We may assume without loss of generality that Q is sharp, and we shall argue by induction on the number of generators of Q. If Q is zero the result is trivial. Suppose that T is a set of nonzero generators for Q, $t \in T$, and $S := T \setminus \{t\}$. Let P be the submonoid of Q generated by S. Then P is still sharp and the induction hypothesis implies that there exists a local homomorphism $h\colon P \to \mathbf{N}$. Then h induces a homomorphism $P^{gp} \to \mathbf{Z}$ which we denote again by h. Replacing h by nh for a suitable $n \in \mathbf{Z}^+$, we may assume that h extends to a homomorphism $Q^{gp} \to \mathbf{Z}$ we which still denote by h. If $h(t) > 0$ there is nothing more to prove. If $h(t) = 0$, choose any $h'\colon Q^{gp} \to \mathbf{Z}$ such that $h'(t) > 0$. Then if n is a sufficiently large natural number, $nh + h'(s) > 0$ for all $s \in S$ and $h'(t) > 0$, so $h \in H(Q)$ and is local. Suppose on the other hand that $h(t) < 0$. For each $s \in S$, let $s' := h(s)t - h(t)s$. Then each $s' \in Q$, and the submonoid Q' of Q generated by the set S' of all s' is sharp. Note that $h(s') = 0$ for all $s' \in S'$ and hence for all $q' \in Q'$. Thus $Q'^{gp} \subseteq \mathrm{Ker}(h) \subseteq Q^{gp}$. Since $|S'| \leq |S|$, the induction hypothesis implies that there exists a local homomorphism $g \in H(Q')$. Replacing g by ng for a suitable n, we may assume that g extends to a homomorphism $\mathrm{Ker}(h^{gp}) \to \mathbf{Z}$, which we continue to denote by g. Since $t \notin \mathrm{Ker}(h^{gp})$, the subgroup of Q^{gp} generated by t and $\mathrm{Ker}\, h$ is isomorphic to $\mathbf{Z} \oplus \operatorname{Ker}(h)$, and we may extend g to this subgroup by letting $g(t) = 0$. Replacing g by yet another multiple, we may assume that it extends to all of Q^{gp}. For any $s \in S$, $-h(t)g(s) = g(s') - h(s)g(t) = g(s') > 0$; since $h(t) < 0$ this implies that $g(s) > 0$. Then $ng - h \in H(Q)$ is local for n sufficiently large. $\square$

Corollary 2.2.3 *Let Q be a fine monoid and let x be an element of Q^{gp}. Then $x \in Q^{\text{sat}}$ if and only if $h(x) \geq 0$ for every $h \in H(Q)$.*

Proof: If $x \in Q^{\text{sat}}$ then $nx \in Q$ for some $n \in \mathbf{Z}^+$ and hence $h(x) \geq 0$ for any $h \in H(Q)$. Suppose conversely that $h(x) \geq 0$ for every $h \in H(Q)$. Let Q' be the submonoid of Q^{gp} generated by Q and $-x$, and choose a local homomorphism $h\colon Q' \to \mathbf{N}$. Then $h(x) \geq 0$ and $h(-x) \geq 0$, so that in fact $h(x) = 0$ and $-x \in Q'^*$. Then there exists an element q' of Q' such that $q' - x = 0$. Writing $q' = -mx + q$ with $m \in \mathbf{N}$ and $q \in Q$, we see that $(m+1)x = q$, so $x \in Q^{\text{sat}}$. $\square$

Proof of (2.2.1) First observe that $H(Q)$ is fine, sharp, and saturated by (2.1.9.8). Since $H(Q) \to \operatorname{Hom}(Q, \mathbf{Z})$ is injective, so is the map $H(Q)^{gp} \to \operatorname{Hom}(Q, \mathbf{Z})$. Any element h of $H(Q)$ necessarily annhilates Q^*, so the image of this map is contained in $\operatorname{Hom}(\overline{Q}, \mathbf{Z})$. Suppose on the other hand that $g \in \operatorname{Hom}(\overline{Q}, \mathbf{Z})$, and let h be a local homomorphism $\overline{Q} \to \mathbf{N}$. There exists $n \in \mathbf{Z}^+$ such that $nh(\overline{q}) \geq g(\overline{q})$ for each of a finite set of nonzero generators $\overline{q}$ of $\overline{Q}$, and then $nh(\overline{q}) \geq g(\overline{q})$ for every $\overline{q} \in \overline{Q}$. This means that $h' := nh - g \in H(\overline{Q})$, so $g = nh - h' \in H(\overline{Q})^{gp} \cong H(Q)^{gp}$. It follows that the map $H(Q)^{gp} \to \operatorname{Hom}(\overline{G}^{gp}, \mathbf{Z})$ is an isomorphism.

Since $H(H(Q))$ is fine saturated and sharp, ev factors through a map $\overline{ev}$ as claimed in the statment of the theorem. Let x_1 and x_2 be two elements of Q^{sat} with $ev(x_1) = ev(x_2)$, and let $x := x_1 - x_2 \in Q^{gp}$. Then $h(x) = 0$ for every $h \in H(Q)$. It follows that from (2.2.3) that x and $-x$ belong to Q^{sat}, so $x \in (Q^{\text{sat}})^*$. Thus $\overline{x}_1 = \overline{x}_2 \in \overline{Q^{\text{sat}}}$, and this proves the injectivity of $\overline{ev}$. For the surjectivity, suppose that $g \in H(H(Q))$. Since Q^{gp} is a finitely generated group, the map from Q^{gp} to its double dual is surjective. Thus there exists an element q of Q^{gp} such that $ev(q) = g$, *i.e.*, such that $h(q) = g(h)$ for all $h \in H(Q)$. Then $h(q) \geq 0$ for all h, so $q \in Q^{\text{sat}}$, as required. $\square$

$\square$

Corollary 2.2.4 *Let Q be a fine monoid. A subset S of Q is a face if and only if there exists an element h of $H(Q)$ such that $S = h^{-1}(0)$. For each $S \subseteq Q$, let $S^\perp$ be the set of $h \in H(Q)$ such that $h(s) = 0$ for all $s \in S$, and for $T \subseteq H(Q)$, let $T^\perp$ be the set of $q \in Q$ such that $t(q) = 0$ for all $t \in T$. Then $F \mapsto F^\perp$ induces an order reversing bijection between the set of faces of Q and the set of faces of $H(Q)$, and $F = (F^\perp)^\perp$ for any face of either.*

Proof: It is clear that $h^{-1}(0)$ is a face of Q if $h \in H(Q)$. If F is any face, Q/F is a fine sharp monoid, so by (2.2.2) there exists a local homomorphism $h\colon Q/F \to \mathbf{N}$. Then h can be regardes as an element of $F^\perp \subseteq H(Q)$. Since h is local, $h^{-1}(0) = F$. This proves the first statement. It is clear that $S^\perp$ is a face of $H(Q)$ if S is any subset of Q and that $T^\perp$ is a face of Q if T is any subset of $H(Q)$. Furthermore, $S_2^\perp \subseteq S_1^\perp$ if $S_1 \subseteq S_2$, and $S \subseteq (S^\perp)^\perp$. The only nontrivial thing to prove is that $F = (F^\perp)^\perp$ if F is a face of Q. But this follows immediately from the existence of an h with $F = h^{-1}(0)$.

$\square$

Corollary 2.2.5 *If Q is fine, then $Q^{\rm sat}$ is again fine. In fact, the action of Q on $Q^{\rm sat}$ defined by the homomorphism $Q \to Q^{\rm sat}$ makes $Q^{\rm sat}$ a finitely generated Q-set.*

Proof: Since $(Q^{\rm sat})^* \subseteq Q^{gp}$, it is a finitely generated abelian group. Theorem (2.2.1) implies that $\overline{Q^{\rm sat}}$ is fine, and since $Q^{\rm sat}$ is integral, it follows from (2.1.1) that $Q^{\rm sat}$ is finitely generated, hence fine. Choose a finite set of generators T for $Q^{\rm sat}$ as a monoid, and for each $t \in T$, choose $n_t \in \mathbf{N}^+$ such that $n_t t \in Q$. Then $\{\sum j_t m_t : j_t \leq n_t, t \in T\}$ generates $M^{\rm sat}$ as a Q-set. $\square$

Corollary 2.2.6 *Let P be a fine sharp monoid such that P^{gp} is torsion free (resp. which is saturated). Then P is isomorphic to a submonoid (resp. an exact submonoid) of $\mathbf{N}^r$ for some r.*

Proof: Note first that if $\pi\colon M \to Q$ is a surjective map of fine monoids, the dual morphism $H(Q) \to H(M)$ is injective and exact. Indeed, we can by (2.2.1) view an element h of $H(Q)^{gp}$ as a homomorphism $\overline{Q} \to \mathbf{Z}$, and we see that $h \in H(Q)$ if and only if $h \circ \pi \in H(M)$. Now let P be a fine sharp monoid such that P^{gp} is torsion free. By (2.1.9.8), $Q := H(P)$ is fine and sharp and $Q^{gp} \cong \mathrm{Hom}(P^{gp}, \mathbf{Z})$, so $P^{gp} \cong \mathrm{Hom}(Q^{gp}, \mathbf{Z}) \cong H(Q)^{gp}$. Choose a surjection $\mathbf{N}^r \to Q$. As we observed above, $H(Q)$ is then an exact submonoid of $H(\mathbf{N}^r) \cong \mathbf{N}^r$. Furthermore, the isomorphism $P^{gp} \to H(Q)^{gp}$ carries P into $H(Q)$, and in fact identifies P^{sat} with $H(Q)$ by (2.2.1). □

Remark 2.2.7 If Q is a fine monoid, then an element h of $H(Q)$ lies in the interior of $H(Q)$ if and only if $h\colon Q \to \mathbf{N}$ is a local homomorphism. Indeed, by definition, an element h of $H(Q)$ belongs to its interior if and only if it is not contained in any proper face of Q. By (2.2.4), this is the case if and only if if $h^\perp$ does not contain any nontrivial face of Q, *i.e.*, if and only if $h^\perp = Q^*$. This is exactly the condition that $h\colon Q \to \mathbf{N}$ be a local homomorphism.

We shall find the following crude finiteness result useful. More more precise variants are available, most of which rely on the theory of Hilbert polynomials in algebraic geometry.

Corollary 2.2.8 *Let Q be a fine sharp monoid of dimension d and let $h\colon Q \to \mathbf{N}$ be a local homomorphism. For each real number r, let*

$$B_h(r) := \{q \in Q : h(q) < r\}.$$

Then there is a constant $c \in \mathbf{R}$ such that for all $r \in \mathbf{R}$,

$$\#B_h(r) < cr^d.$$

Proof: By (2.2.1), $H(Q)$ is finitely generated and sharp, and hence it has a unique set of minimal generators $\{h_1, \dots h_m\}$. Since h is local, (2.2.7) shows that each h_i belongs to the face generated by h. Then (1.3.2) implies that for each i there exists an integer n_i such that $n_i h \geq h_i$ in $H(Q)$. Choose $n \geq n_i$ for all i. Then for every $r \in \mathbf{R}^+$, $B_h(r) \subseteq \cap_i B_{h_i}(nr)$. Since Q is sharp, (2.2.1) implies that $H(Q)^{gp} \cong \mathrm{Hom}(Q^{gp}, \mathbf{Z})$, and consequently $\{h_i\}$ spans $\mathrm{Hom}(Q^{gp}, \mathbf{Z})$. Proposition 1.3.6 says that this group has rank d. Let $(x_1, \cdots x_d)$ be a basis for $\mathrm{Hom}(Q^{gp}, \mathbf{Z})$, find integers $a_{i,j}$ such that $x_i = \sum_j a_{i,j} h_j$, and let $a := \sum_{i,j} |a_{i,j}|$. Then if $q \in B_h(r)$,

$$|x_i(q)| \leq \sum_j |a_{i,j}| h_j(q) \leq anr.$$

Thus $B_h(r) \subseteq \cap_i B_{|x_i|}(anr)$. The cardinality of this set is bounded by $t(2anr)^d$, where t is the order of the torsion subgroup of Q^{gp}. □

1

2.3 Monoids and cones

Let K be an Archimidean ordered field and let $K^{\geq 0}$ denote the set of nonnegative elements of K, regarded as a multiplicative monoid. Since $0 \in K^{\geq 0}$, this monoid is not quasi-integral, but $K^{\geq 0} \setminus \{0\}$ is a group. In practice here, K will be either $\mathbf{R}$ or $\mathbf{Q}$.

Definition 2.3.1 *A K-cone is an integral monoid $(C, +, 0)$ endowed with an action of $(K^{\geq 0}, \cdot, 1)$, such that*

$$\begin{aligned} (a+b)x &= ax + bx \quad \text{for } a, b \in K^{\geq 0} \text{ and } x \in C, \text{ and} \\ a(x+y) &= ax + ay \quad \text{for } a \in K^{\geq 0} \text{ and } x, y \in C. \end{aligned}$$

A morphism of K-cones is a morphism of monoids compatible with the actions of $K^{\geq 0}$.

Any K-vector space V forms a K-cone, and any nonempty subset of C of V which is stable under addition and by multiplication $K^{\geq 0}$ is a subcone. If C is any K-cone, then C^{gp} inherits a unique structure of a K-vector space such that $C \to C^{gp}$ is a morphism of K-cones, so we can regard every K-cone as sitting inside a K-vector space. If S is any subset of a K-vector space V we can define its *conical hull* $C_K(S)$ to be the set of all linear combinations of elements of S with coefficients in $K^{\geq 0}$. Then $C_K(S)$ is the smallest K-cone in V containing S. A K-cone C is called *finitely generated* if it admits a finite subset S such that $C = C_K(S)$. In the sequel we shall say "cone" instead of "K-cone," and write $C(S)$ instead of $C_K(S)$, when there seems to be no danger of confusion.

If C is a K-cone, C^* is not just a subgroup but also a vector subspace, the largest linear subspace of C. A cone is sharp if and only if$C^* = 0$; some authors call such a C a *strongly convex cone.* If C is a K-cone, then $\overline{C} := C/C^*$ is a sharp K-cone. By the *dimension* of C we mean the dimension of C^{gp} (as a K-vector space), and we call the dimension of $\overline{C}$ the *sharp dimension* of C.

Let C be a K-cone and let F be a face of C. Then F is automatically a subcone of C. Indeed, if $x \in F$ and $a \in K^{\geq 0}$, then there exists $n \in \mathbf{N}$with $a \leq n$, since K is Archimidean. Then $ax \leq nx$ and $nx \in F$, and since F is a face, $ax \in F$ also. If F is a face of a cone C, then C/F is a sharp cone, and we call its dimension the *codimension of F*. If this codimension is one, we say that F is a *facet* of C. A one-dimensional face of C is sometimes called an *extremal ray* of C.

Let us say that an element x of a sharp cone C is *K-indecomposable in C* if it is not a unit and whenever $x = y + z$ with y and z in C, then y and z are K-multiples of x. Thus x is K-indecomposable if and only $\langle x \rangle^{gp}$ is a one-dimensional K-vector space. Notice that in the monoid P given by generators $\{x, y, z\}$ and relations $x + y = 2z$, x, y, and z are irreducible, and in the corresponding cone x and y are indecomposible, but z is not indecomposible.

Proposition 2.3.2 *Suppose that C is a finitely generated sharp cone. Then each element of every minimal set of generators for C is K-indecomposable. In particular, C is spanned by its indecomposable elements.*

Proof: The proof is essentially the same as the proof of the analogous result (2.1.2) for monoids, but we write it in detail anyway. Suppose that S is a minimal set of generators and $x \in S$. Write $x = y + z$, with $y = \sum a_s s$, $z = \sum b_s s$, and $a_s, b_s \in K^{\geq 0}$. Then $x = \sum c_s s$, with $c_s = a_s + b_s$. Let $S' := S \setminus \{x\}$, so $(1 - c_x)x = \sum_{s \in S'} c_s s$. If $c_x < 1$ we see that S' generates C, a contradiction, and if $c_x > 1$, then x is a unit, contradicting the sharpness of C. Write $y = y' + a_x x, z = z' + b_x x$, so that $x = y + z = y' + z' + a_x x + b_x x = y' + z' + x$. Thus $y' + z' = 0$, hence $y' = z' = 0$, so $y = a_x x$ and $z = b_z x$. $\square$

Proposition 2.3.3 *Let C be a finitely generated cone and S a finite set of generators for C.*

1. *Every face of C is generated as a cone by $F \cap S$.*
2. *C contains only a finite number of faces.*
3. *Every inclusion $F \subseteq F'$ of faces of C is contained in a strictly increasing chain $C^* = F_0 \subset F_1 \subset F_2 \cdots F_d = C$ of faces with d equal to the dimension of the K-vector space $\overline{C}^{gp}$, and no such chain has length greater than $\dim \overline{C}^{gp}$.*
4. *If $\overline{C}^{gp} \neq 0$, every proper face of C is contained in a facet.*

Proof: Let F be a face of C and $x \in F$, $x \neq 0$. Then we can write $x = \sum a_s s$ with $a_s \in K^{\geq 0}$ and $s \in S$, Since F is a face, each $s \in F$ if $a_s \neq 0$. This shows that in fact F is generated by the finite set $F \cap S$. Since

S has only finitely many subsets, C can have only finitely many faces. We prove (3) by induction on the dimension $\overline{d}$ of $\overline{C}^{gp}$. Since there is a natural bijection between the faces of C and the faces of $\overline{C}$ we may as well assume that $C^* = 0$. If $\overline{d} = 0$, $C = 0$ and the result is trivial. Suppose that $\overline{d} > 0$; we may assume by (2.3.2) that S is the set of indecomposable elements of C. Let $F \subseteq F'$ be faces of C and let $\mathcal{C} := F_0 \subset \cdots F_d = C$ be a maximal chain containing F and F'. Then $d \geq 1$, and since $F_1 \neq 0$, by (1) it must contain a K-indecomposable element c. Then $\langle c \rangle \subseteq F_1$, and since $\mathcal{C}$ is a maximal chain, $\langle c \rangle = F_1$. Since c is K-indecomposable, $\langle c \rangle^{gp}$ is a one-dimensional K-vector space, and the dimension of $(C/F_1)^{gp} \cong C^{gp}/F_1^{gp}$ is $\overline{d} - 1$. Since the image of $F_1 \subset F_2 \subset \cdots F_d$ in C/F_1 is a saturated chain in C/F_1, it follows from the induction hypothesis that $d = \overline{d}$. This proves (3), and (4) is an immediate consequence. ☐

Proposition 2.3.4 *The interior (i.e., the complement of the union of the proper faces) of a finitely generated cone C is dense in C (in the standard topology).*

Proof: We may and shall assume without loss of generality that C is sharp. Let S be the set of indecomposable elements of C. Then any element c of C can be written (not uniquely) as $c = \sum a_s s$ with $a_s \geq 0$, and c lies in the interior if no $a_s = 0$. Then $c_i := \sum (a_s + i^{-1}) s$ lies in the interior of C and converges to c. ☐

Let P be an integral monoid and consider the map $P \to K \otimes P^{gp}$ sending an element p to $1 \otimes p$. Let $C_K(P)$ denote the subcone of $K \otimes P^{gp}$ generated by the image of $P \to K \otimes P^{gp}$, and

$$c \colon P \to C_K(P)$$

be the map sending $p \in P$ to $1 \otimes p \in C_K(P)$. Note that two elemets p_1 and p_2 of P have the same image in $K \otimes P^{gp}$ if and only if their difference lies in the torsion subgroup of P^{gp}, *i.e.*, iff there exists an integer n, such that $np_1 = np_2$.

Proposition 2.3.5 *Let P be an integral monoid and let $\gamma \colon P \to C_K(P)$ be the natural map described above.*

1. *Then if* F *is any face of* P*, the natural map* $C_{\mathbf{Q}}(F) \to C_{\mathbf{Q}}(P)$ *identifies* $C_{\mathbf{Q}}(F)$ *with a face of* $C_{\mathbf{Q}}(P)$*. Furthermore,* $\gamma^{-1}(C_{\mathbf{Q}}(F)) = F$*, and* γ *defines a bijection between the faces of* $C_{\mathbf{Q}}(P)$ *and the faces of* F*.*
2. *If* I *is an ideal of* P*, let* $C_{\mathbf{Q}}(I) \subseteq C_{\mathbf{Q}}(P)$ *denote the smallest* $\mathbf{Q}^{\geq 0}$*-invariant ideal of* $C_{\mathbf{Q}}(P)$ *containing the image of* $I \to C_{\mathbf{Q}}(P)$*. Then* $C_{\mathbf{Q}}(I) \cap P = \sqrt{I}$*.*

Proof: The proof relies on the following lemma, which is not true for a general K. However, see Proposition (2.3.13) for a partial generalization of Proposition (2.3.5).

Lemma 2.3.6 *Let* P *be a monoid and let* $C_{\mathbf{Q}}(P) \subseteq \mathbf{Q} \otimes P^{gp}$ *the corresponding cone. Then*

$$C_{\mathbf{Q}}(P) = \{x \in \mathbf{Q} \otimes P^{gp} : \text{ there exist } m \in \mathbf{Z}^+, p \in P \text{ with } mx = c(p).\}$$

If I *is an ideal of* P*,*

$$C_{\mathbf{Q}}(I) = \{x \in \mathbf{Q} \otimes P^{gp} : \text{ there exist } m \in \mathbf{Z}^+, p \in I \text{ with } mx = c(p).\}$$

Proof: If $m_1 x_1 = c(p_1)$ and $m_2 x_2 = c(p_2)$, then

$$m_1 m_2 (x_1 + x_2) = c(m_2 p_1 + m_1 p_2),$$

so the set X on the right side of the above equation is a submonoid of $\mathbf{Q} \otimes P^{gp}$. It is also stable under the action of $\mathbf{Q}^{\geq 0}$ and contains the image of P, hence contains $C_{\mathbf{Q}}(P)$. On the other hand, it is also clear that X is contained in any $\mathbf{Q}$-cone containing the image of P, hence is the smallest such cone. □

Now let F be a face of P and let x_1 and x_2 be elements of $C_{\mathbf{Q}}(P)$ whose sum y belongs to $C_{\mathbf{Q}}(F)$. Then there exist $m > 0$, $f \in F$ and $p_i \in P$ such that $my = 1 \otimes f$ and $mp_i = 1 \otimes x_i$. Hence $f - p_1 - p_2$ is a torsion element of P^{gp}, and by replacing m by a multiple, m we may assume that $f = p_1 + p_2$. Then $p_i \in F$ and hence $x_i \in C_{\mathbf{Q}}(F)$. This shows that $C_{\mathbf{Q}}(F)$ is a face of $C_{\mathbf{Q}}(P)$. Evidently $F \subseteq c^{-1}(C_{\mathbf{Q}}(F))$. Conversely, if $p \in P$ and $c(p) \in C_{\mathbf{Q}}(F)$, then there exist an $m \in \mathbf{Z}^+$ and $f \in F$ with $c(f) = mc(p)$, hence there exist m' such that $m'f = mm'p \in P$, and hence $p \in F$. On the other hand, if G is any face of $C_{\mathbf{Q}}(P)$ and g is a generator for G as a face, then mg lies in the image of c for some m, and mg still generates G. Thus $G = C_{\mathbf{Q}}(F)$, where $F := c^{-1}(G)$. This proves (1), and the proof of (2) is similar. □

Corollary 2.3.7 *If Q is a fine monoid, the map $Q \to Q^{\mathrm{sat}}$ induces a homeomorphism* $\operatorname{Spec}(Q^{\mathrm{sat}}) \to \operatorname{Spec}(Q)$.

Proof of (1.3.6) Because of the bijection between the prime ideals and the faces of M and the bijection (2.3.5) between the faces of M and of the cone C it spans, (1.3.6) follows from (2.3.3). Thus, M has finitely many prime ideals because C has finitely many faces, and the maximal length of a chain of prime ideals in M is the maximal length of a chain of faces of C. By (2.3.5) this is the dimension of the vector space $\overline{C}^{gp} \cong \mathbf{Q} \otimes \overline{M}^{gp}$. If $\mathfrak{p} \in \operatorname{Spec} M$, and $F_{\mathfrak{p}} = M \setminus \mathfrak{p}$ is the corresponding face of C, then by (2.3.3.3), $F_{\mathfrak{p}}$ is contained in a chain of length $\dim(\overline{C}) = \dim(M)$. Furthermore $\operatorname{ht}(\mathfrak{p})$ is by definition the maximum length h of a chain of faces $F_{\mathfrak{p}} = F_0 \subset F_1 \cdots F_h = C$, *i.e.*, of a chain of faces in $C/F_{\mathfrak{p}}$. By (2.3.3.3), $h = \dim(C/F_{\mathfrak{p}})^{gp} = \dim(\overline{C}^{gp}) - \dim(F_{\mathfrak{p}}^{gp})$, so $h + \dim(F_{\mathfrak{p}}) = \dim(M)$. □

Corollary 2.3.8 *Let $\mathfrak{p}$ be a height one prime ideal in a fine monoid M. Then $M_{\mathfrak{p}}^{\mathrm{sat}}$ is valuative, and there is a unique isomorphism*

$$\overline{M_{\mathfrak{p}}^{\mathrm{sat}}} \cong \mathbf{N},$$

and a unique epimorphism

$$\nu_{\mathfrak{p}} \colon M^{gp} \to \mathbf{Z}$$

such that $\nu_{\mathfrak{p}}^{-1}(\mathbf{N}^+) \cap M = \mathfrak{p}$. Furthermore, $M_{\mathfrak{p}}^{\mathrm{sat}} = \{x \in M^{gp} : \nu_{\mathfrak{p}}(x) \geq 0\}$

Proof: We know that M^{sat} is fine, $M^{gp} \cong (M^{\mathrm{sat}})^{gp}$, and that $\operatorname{Spec}(M^{\mathrm{sat}}) \to \operatorname{Spec}(M)$ is a homeomorphism. Thus we may as well assume replace M by M^{sat}, and so we assume that M is saturated. Since $M_{\mathfrak{p}}$ is saturated, $\overline{M_{\mathfrak{p}}}^{gp}$ is torsion free, and since $\mathfrak{p}$ has height one, $\overline{M_{\mathfrak{p}}}^{gp}$ is isomorphic to $\mathbf{Z}$. Choose any nonzero element x of $\overline{M_{\mathfrak{p}}}$. Then there is an $n \in \mathbf{N}^+$ such that $x = ny$, where y is one of the two generators of $\overline{M_{\mathfrak{p}}}^{gp}$. Since $\overline{M_{\mathfrak{p}}}$ is saturated, $y \in \overline{M_{\mathfrak{p}}}$, and y freely generates $\overline{M_{\mathfrak{p}}}$. This shows that $M_{\mathfrak{p}}$ is saturated. Furthermore, $-y \notin \overline{M_{\mathfrak{p}}}$, so the induced isomorphism $\overline{M_{\mathfrak{p}}} \to \mathbf{N}$ is unique. Let μ be the composition $M \to \overline{M_{\mathfrak{p}}}^{gp} \to \mathbf{N}$, then $\mu^{-1}(\mathbf{N}^+) = \mathfrak{p}$, and $\nu_{\mathfrak{p}} := \mu^{gp}$ is an epimorphism such that $\nu_{\mathfrak{p}}^{-1}(\mathbf{N}^+) \cap M = \mathfrak{p}$. Suppose that $\nu \colon M^{gp} \to \mathbf{Z}$ is an epimorphism such that $\nu^{-1}(\mathbf{N}^+) \cap M = \mathfrak{p}$. Then $\nu^{-1}(0) \cap M$ is the face $F := M \setminus \mathfrak{p}$, and ν factors through $\overline{M_{\mathfrak{p}}}^{gp} \cong \mathbf{Z}$. Since ν is an epimorphism, this last map is an isomorphism, and $\nu = \pm\nu_{\mathfrak{p}}$. In fact the sign must be + since $\nu^{-1}(\mathbf{N}^+) = \mathfrak{p}$. If q and p are elements of M, $\nu_{\mathfrak{p}}(p - q) = \nu_{\mathfrak{p}}(p) - \nu_{\mathfrak{p}}(q)$. Thus if $q \in M \setminus \mathfrak{p}$, $\nu_{\mathfrak{p}}(q) = 0$ and $\nu_{\mathfrak{p}}(p - q) \geq 0$. Conversely, if $x \in M^{gp}$ and $\nu_{\mathfrak{p}}(x) \geq 0$, there exists a $q \in M_{\mathfrak{p}}$ such that $\nu_{\mathfrak{p}}(q) = \nu_{\mathfrak{p}}(x)$. Then there exists $u \in M^*_{\mathfrak{p}}$ such that $x = q + u$, and $x \in M_{\mathfrak{p}}$. □

Corollary 2.3.9 *Let Q be a fine saturated monoid. Then $Q = \{x \in Q^{gp} : \nu_{\mathfrak{p}}(x) \geq 0\}$, where $\mathfrak{p}$ ranges over the set of height one primes of Q (2.3.8). In other words, Q is the intersection in Q^{gp} of the set of all its localizations at height one primes.*

Proof: We know from (2.1.9.8) that $H(Q)$ is a fine sharp monoid, and from (2.3.2) that the $\mathbf{Q}$-cone C it generates is generated by a finite set $(h_1, \cdots h_n)$ of indecomposible elements. Each h_i generates a one dimensional face of C; consequently each $h_i^{\perp}$ is a facet of Q, and $\mathfrak{p}_i := h_i^{-1}(\mathbf{N}^+)$ is a height one prime of Q. If $x \in Q_{\mathfrak{p}}$ for every height one prime $\mathfrak{p}$, then $h_i(x) \geq 0$ for every i and hence $h(x) \geq 0$ for every $h \in C$, and hence for every $h \in H(Q)$. Then $x \in Q$ by (2.2.1) □

A cone is called *simplicial* if it is finitely generated and free, that is, if there exists a finite set S such that each element of C can be written uniquely as a linear combination of elements $\sum_{s \in S} a_s s$ with $a_s \in K^{\geq 0}$; such a set S necessarily forms a basis for C^{gp}. It is not hard to see that any sharp cone in K^1 or K^2 is simplicial. This is false for K^3; for example, the cone generated by the monoid P of (1.3.7) is not simplicial. For a useful criterion, see (2.3.14) below.

In fact, every finitely generated cone is a finite union of simplicial cones, as the following result of Carathéodory shows.

Theorem 2.3.10 (Carathéodory) *Let C be a K-cone and let S be a set of generators for C. Then every element of C lies in a cone generated by a linearly independent subset of S.*

Proof: If $x \in C$, we can write $x = \sum a_i s_i$ with $s_i \in S$ and $a_i > 0$. We may suppose that this has been done so that the number e of terms in the sum is minimal, and we claim that then $(s_1, s_2, \ldots s_e)$ is independent in C^{gp}. In fact suppose that $\sum c_i s_i = 0$. We may choose the indexing so that c_i is positive if $1 \leq i \leq m$, negative if $m < i \leq n$, and zero if $i > n$. Furthermore, we may suppose that $a_1/c_1 \leq a_2/c_2 \cdots \leq a_m/c_m$ and that $a_n/c_n \geq a_{n-1}/c_{n-1} \cdots \geq a_{m+1}/c_{m+1}$. Suppose $m > 0$. Then for all i, $a'_i := a_i - (a_1/c_1)c_i \geq 0$, and then $xx - (a_1/c_1)\sum c_i s_i = \sum\{a'_i s_i : i > 1\}$, contradicting the minimality of e. Thus $m = 0$. If $n > 0$, then for all i $a'_i = a_i - (a_n/c_n)c_i \geq 0$, and $x = \sum\{a'_i : i \neq n\}$, again a contradiction. Thus $n = 0$, all $c_i = 0$, and $(s_1, \ldots s_e)$ is linearly independent. □

Corollary 2.3.11 *Let C be a finitely generated sharp K-cone of dimension d. Then C is a finite union of simplicial cones of dimension d.*

Proof: Let S be a finite set of generators of C. Since the K-span of S is C^{gp}, whose dimension is d, any linearly independent subset T of S is contained in a linearly independent subset T' of cardinality d. Carathéodory's theorem implies that every element of c belongs to some $C(T)$ and hence to some $C(T')$. □

Corollary 2.3.12 *Let C be be a finitely generated cone in a finite dimensional K-vector space V. Then C is closed with respect to the topology of V induced from the ordering on K. In particular, any face of a finitely generated cone C is closed in C, and the interior I_C (1.3) of C is open.*

Proof: The group C^* of units of C is a K-subspace of V, hence is closed, and hence it suffices to prove that the image of C in V/C^* is closed. Thus we may and shall assume that C is sharp. Suppose that V has dimension n and that C is simplicial of dimension d. Then there exists a basis $(v_1, \ldots v_n)$ for V such that $(v_1, \ldots v_d)$ spans C. Thus V can be identified with K^n and C with the subset of $v \in K^n$ such that $v_i \geq 0$ for $i \leq d$ and $v_i = 0$ for $i > d$. Since the topology on V is independent of the choice of basis, C is closed. The general case follows, since Corollary (2.3.11) shows that any C can be written as a finite union of simplicial cones. Finally we recall from (2.3.3) that a face of a finitely generated cone is finitely generated, hence closed. Since C has only a finite number of faces, I_C is open. □

Proposition 2.3.13 *Let C be a finitely generated $\mathbf{Q}$-cone and let $C_K \subseteq K \otimes C^{gp}$ be the K-cone it spans.*

1. *For every $x \in C_K$ there exists an increasing sequence $(x_i : i \in \mathbf{N})$ in C converging to x. In particular, C is dense in C_K.*
2. *If C' is any finitely generated subcone of C, $C'_K \cap C^{gp} = C'$.*
3. *The map $F \mapsto F_K$ induces a bijection between the faces of C and the faces of C_K, with inverse $G \mapsto G \cap C$.*

Proof: Let S be a finite set of generators for C; then S also generates C_K as an $\mathbf{R}$-cone. Any element x of C_K can be written $x = \sum a_s s$ with $a_s \in \mathbf{R}^{\geq 0}$. For each s there exists an increasing sequence a_{is} in $\mathbf{Q}^{\geq 0}$ converging to a_s; then $x_i := \sum a_{is} s$ is an increasing sequence in C converging to x. This proves (1). To prove (2), suppose that T is a finite set of generators for C' and $x' \in C'_K$. By (2.3.10) there exist a linearly independent subset T' of T and elements $a_t \in \mathbf{R}^{\geq 0}$ such that $x' = \sum\{a_t t : t \in T'\}$. Since C spans C^{gp}, there is a basis S' for C^{gp} which contains T' and is contained in C. If $x' \in C'_K \cap C^{gp}$, all its coordinates with respect to S' lie in $\mathbf{Q}$. In particular each $a_t \in \mathbf{Q}^{\geq 0}$, so $x' \in C'$. This proves (2).

Now suppose F is a face of C. It is clear from the definition that F_K is a submonoid of C_K; to prove that it is a face we must check that if $x \leq y$ with $x \in C_K$ and $y \in F_K$, then $x \in F_K$. Recall that F is generated as a cone by $F \cap K$, so y can be written $y = \sum_s a_s s$ with $a_s \in K^{\geq 0}$ and $s \in F \cap S$. Replacing by $\sum_s a'_s s$ with $a'_s \in \mathbf{Q}$ and $a'_s \geq a_s$, we may assume that $y \in F$. By (1) we can find an increasing sequence (x_i) in C converging to x. For each fixed j, $(x_i - x_j : i \in \mathbf{N})$ is a sequence in C^{gp} which converges to $x - x_j$ and for $i > j$ lies in C; since C_K is closed it follows that $x - x_j \in C_K$ also. Then $y - x_j = y - x + x - x_j \in C_K \cap C^{gp} = C$, and since F is a face of C, $x_j \in F$ for all j. By (2.3.2) F is a finitely generated as a cone and so F_K is closed in C_K. Hence $x \in F_K$ as required. The fact that $F_K \cap C^{gp} = F$ follows from (2). Finally, if G is any face of C_K, we know from (2.3.2) that G is generated by a subset of S, hence by $G \cap C$, which is a face of C. □

Proposition 2.3.14 *Let C be a sharp K-cone and S a subset of C. Suppose that C is a finitely generated sharp K-cone and $S \subseteq C$ a finite subset of C which spans C^{gp} as a K-vector space. Then S is linearly independent and spans C a K-cone. In particular, C is then simplicial.*

Proof: Suppose that $\sum a_s s = 0$ with $a_s \in K$ and $s \in S$. Let $S' := \{s \in S : a_s > 0\}$, $S'' := \{s \in S : a_s < 0\}$, and $T := S \setminus S' \cup S''$. Then let t be the sum of all the elements of T, and let

$$f := \sum_{s \in S'} a_s s + t = \sum_{s \in S''} -a_s s + t;$$

note that $f \in C$. If S'' is not empty, then $S' \cup T$ is a proper subset of S and hence by assumption is contained in a proper face F of C. Since $f = \sum\{-a_s s : s \in S''\} \in F$ and F is a face, all the elements of S'' also belong to F. But then all of S is contained in F. Then $F^{gp} = C^{gp}$ and since F is exact in C, $F = C$, a contradiction. Thus we must have $S'' = \emptyset$. Similarly $S' = \emptyset$, and it follows that S is linearly independent.

Let c be an element of the interior of C. Then there exist disjoint subsets S' and S'' of S and elements $a_s \in K^{\geq 0}$ such that

$$c = \sum_{s \in S'} a_s s - \sum_{s \in S''} a_s s.$$

Then $c + \sum\{a_s s : s \in S''\}$ also belongs to the interior of C. If S' were a proper subset of S, it would be contained in a proper face of C, which contradicts the fact that $\sum\{a_s s : s \in S'\} = c + \sum\{a_s s : s \in S''\}$ is in the interior of C. Hence $S' = S$ and $S'' = \emptyset$. We have thus shown that every element of the interior of C lies in the the $K^{\geq 0}$-span of S. Since this span is closed, and since the interior of C is dense in C (2.3.4), S spans C, as claimed. $\square$

Theorem 2.3.15 (Gordon's lemma) *Let L be a finitely generated abelian group, let $V := \mathbf{Q} \otimes L$, and let $C \subseteq \mathbf{Q} \otimes L$ be a finitely generated $\mathbf{Q}$-cone. Then $C_L := L \times_V C \cong L \times_{V_{\mathbf{R}}} C_{\mathbf{R}}$ is a finitely generated monoid.*

Proof: The natural map $L \times_V C \to L \times_{V_{\mathbf{R}}} C_{\mathbf{R}}$ is injective because $C \subseteq C_{\mathbf{R}}$, and it is surjective because of (2.3.13.2). Let us first treat the case in which L is free, so that it may be identified with its image in V. Let S be a finite set of generators for C, which we may as well assume contained in L. Let $S' \subseteq V_{\mathbf{R}}$ be the set of all linear combinations of elements of S with coefficients in the interval $[0,1]$. The map $[0,1]^S \to V_{\mathbf{R}}$ sending $\{a_s : s \in S\}$ to $\sum a_s s$ is continuous and maps surjectively to S'; hence S' is compact. Then $S'' := L \cap S'$ is compact and discrete, hence finite. Any element x of $C_{\mathbf{R}}$ can be written as a sum $\sum a_s s$ with $s \in S$ and $a_s \in \mathbf{R}^{\geq 0}$, and a_s can be written $a_s = m_s + a'_s$ with $m_s \in \mathbf{N}$ and $a'_s \in [0, 1]$. Then $x = \sum m_s s + s'$ with $s' \in S'$; if also $x \in L$, in fact $s' \in S''$, and so x is a sum of elements of S''. Thus the monoid $C_L = L \cap C_{\mathbf{R}}$ is generated by the finite set S''. For the general case, let L_t be the torsion subgroup of L and let $L_f := L/L_t$. Notice that $L_t \subseteq C^*_L$, and the natural map $C_L \to L$ identifies C_L/L_t with $C_{L_f} = L_f \cap C$ and C^*_L/L_t with $C^*_{L_f}$. Since C_{L_f} is a fine monoid, it follows from (2.1.1) that $C^*_{L_f}$, is a finitely generated group, and since L_t is finitely generated, so is C^*_L. Now (2.1.1) implies that C_L is a finitely generated monoid. $\square$

The finiteness of the saturation of a fine monoid also follows from Gordon's lemma.

Corollary 2.3.16 *Let M be a fine monoid and let $C \subseteq K \otimes M^{gp}$ be the K-cone it spans. Then $M^{\mathrm{sat}} = M^{gp} \times_{C^{gp}} C$ and is finitely generated as a monoid.*

Proof: The previous result implies that $M^{gp} \times_{C^{gp}} C$ is finitely generated as a monoid and is independent of the choice of K, so we may as well take $K = \mathbf{Q}$. If $x \in M^{\mathrm{sat}}$, then by definition $x \in M^{gp}$ and there exists $n > 0$ such that $nx \in M$. It follows that $1 \otimes x = (1/n)(1 \otimes nx)$ lies in C, so $x \in M^{gp} \times_{C^{gp}} C$. Conversely, if $x \in M^{gp}$ and $1 \otimes x \in C$, then there exist $x_i \in M$ and $a_i \in \mathbf{Q}^{\geq 0}$ such that $1 \otimes x = \sum a_i(1 \otimes x_i)$. Choose $n \in \mathbf{N}^+$ such that $na_i \in \mathbf{N}$ for all i. Then $1 \otimes nx = 1 \otimes y$ where $y := \sum na_i x_i \in M$. Thus $nx = y + z$ with $y \in M$ and $z \in M^{gp}_t$. If $m \in \mathbf{N}^+$ is such that $mz = 0$, then $mnx = my$, so $x \in M^{\mathrm{sat}}$. We conclude that M^{sat} is finitely generated as a monoid. $\square$

2.4 Idealized monoids

A surjective map of commutative rings $A \to B$ induces a closed immersion $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, but the analog for monoid is not true: if $Q \to P$ is any morphism of monoids, the generic point of $\mathrm{Spec}\, Q$ lies in the image of $\mathrm{Spec}\, P$, so the map $\mathrm{Spec}\, P \to \mathrm{Spec}\, Q$ cannot be a closed immersion unless it is bijective. To remedy this we introduce the category **Imon** of *idealized monoids.* This is the category of pairs (Q, J), where Q is a monoid and J is an ideal of Q; morphisms $(Q, J) \to (P, I)$ are morphisms $Q \to P$ sending J

to I. The functor **Imon** $\to$ **Mon** taking (Q, J) to Q has a left adjoint, taking a monoid P to $(P, \emptyset)$, and we can view **Mon** as a full subcategory of **Imon**. Furthermore we have a functor from the category of commutative rings to the category **Imon**, taking a ring A to its multiplicative monoid together with the zero ideal.

If I is an ideal of a monoid Q, then the ideal of $R[Q]$ generated by $e(I)$ is free with basis $e_{|I}$, and we denote it by $R[I]$. Thus the quotient $R[Q]/R[I]$ is a free R-module with basis $Q \setminus I$. For any R-algebra A, $\mathrm{Hom}_{\mathbf{Imon}}((Q, I), (A, 0)) = \mathrm{Hom}_R(R[Q]/R[I], A)$, so that the functor $(Q, I) \mapsto R[Q]/R[I]$ is left adjoint to the functor $A \mapsto (A, 0)$.

Inductive and projective limits exist in the category of idealized monoids, and are compatible with the forgetful functor **Imon** $\to$ **Mon**. For example, if $u_i\colon (P, I) \to (Q_i, J_i)$ is a pair of morphisms and $v_i\colon Q_i \to Q$ is the pushout of the underlying monoid morphisms, then $v_i\colon (Q_i, J_i) \to (Q, J)$ is the pushout, where J is the ideal of Q generated by the images of J_i.

A morphism $\theta\colon (Q, J) \to (P, I)$ is *ideally strict* if I is generated by the image of J, and is *strict* if in addition its underlying morphism is strict. Note that θ is ideally strict if and only if $\overline{\theta}$ is. We say that θ is *ideally exact* if $J = \theta^{-1}(I)$, and that it is *exact* if in addition its underlying morphism is exact. Note that if the underlying morphism of θ is strict, then $\overline{\theta}$ is bijective, and hence θ is ideally strict if and only if it is ideally exact.