Math 185 Problem Set 4

Problem 1. (20: 5,5,5,5 points)
(a) Use Cauchy's integral formula to compute

$$\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)}\, dz,$$

where C is the circle of radius 4: $|z| = 4$.

(b) Compute $\displaystyle\int_C \frac{z^2}{z^2+1}\, dz$, where C is the circle of radius 1 centered at $z = i$.

(c) Let C be the circle of radius 2: $|z| = 2$. Use Cauchy's integral formula to compute

$$\int_C \frac{\overline{z}}{z^2-1}\, dz.$$

Be careful: $\overline{z}$ is not analytic, but there is a way around this.

(d) Let $\theta = \arg(z)$ Take C to be the wavy contour in the z-plane described by $0 \le \arg(z) \le \pi$; $|z| = 1 - 0.1\cos(100\theta)$. Compute the integral $\displaystyle\int_C z^2\, dz$.

Problem 2. (15: 10,5 points)
(a) Let $f(z) = z^n$, where n is a positive integer. By directly computing the integral, show that Cauchy's integral formula holds for $f(z_0)$ and Cauchy's formula for derivatives holds for $f'(z_0)$.

You may need the binomial formula for expanding $(a+b)^n$. As a hint: you may want to make a short argument, based on Cauchy's theorem, reducing the integrals to circles centered on the point of interest.

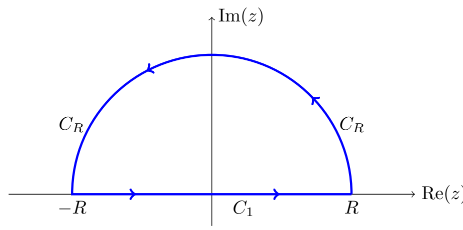
(b) Let $P(z) = c_o + c_1 z + c_2 z^2 + c_3 z^3$. Let C be the circle $|z| = a$, for $a > 0$. Compute the integral

$$\int_C P(z) z^{-n} dz \ \text{ for n = 0, 1, 2, } \ldots$$

Problem 3. (15: 5,5,5 points)
(a) Compute $\displaystyle\int_C \frac{|z|e^z}{z^2}\, dz$ where C is the circle $|z| = 2$.

(b) Compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx$.

Hint: integrate over the closed path shown below. Show that as R goes to infinity the contribution of the integral over C_R becomes 0.



(c) Show that $\displaystyle\int_{|z|=2} \frac{1}{z^2(z-1)^3}\,dz = 0$ in two different ways.

(i) Use Cauchy's integral formula. You'll need to divide the contour to isolate each of the singularities of the integrand.

(ii) First, show that the integral doesn't change if you replace the contour by the curve $|z| = R$ for $R > 2$. Next, show that this integral must go to 0 as R goes to infinity.

Problem 4. (5 points)
Suppose f is analytic on and inside a simple closed curve C. Assume $f(z) = 0$ for z on C. Show $f(z) = 0$ for all z inside C.

Problem 5. (10 points)
Let γ be a simple closed curve that goes through the point $1+i$. Let

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{\cos(w)}{w-z}\,dw.$$

Find the following limits:
(i) $\displaystyle\lim_{z\to 1+i} f(z)$, where z goes to $1+i$ from outside γ.

(ii) $\displaystyle\lim_{z\to 1+i} f(z)$, where z goes to $1+i$ from inside γ.

Problem 6. (10: 5,5 points)
(a) Suppose that $f(z)$ is analytic on a region A that contains the disk $|z - z_0| \le r$. Use Cauchy's integral formula to prove the mean value property

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta.$$

(b) Prove the more general formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi r^n}\int_0^{2\pi} f(z_0 + re^{i\theta})e^{-in\theta}\,d\theta.$$

Problem 7. (20: 4,4,4,4,4 points)
Let C be the curve $|z| = 2$. Explain why each of the following integrals is 0.

(a) $\displaystyle\int_C \frac{z}{z^2+35}\,dz.$

(b) $\displaystyle\int_C \frac{\cos(z)}{z^2-6z+10}\,dz.$

(c) $\displaystyle\int_C \mathrm{e}^{-z}(2z+1)\,dz.$

(d) $\displaystyle\int_C \log(z+3)\,dz$ (principal branch of log).

(e) $\displaystyle\int_C \sec(z/2)\,dz.$

Extra problems not to be scored.

Problem 8. (0 points)
Show $\displaystyle\int_0^{\pi} \mathrm{e}^{\cos\theta}\cos(\sin(\theta))\,d\theta = \pi$. Hint, consider e^z/z over the unit circle.

Problem 9. (0 points)
(a) Suppose $f(z)$ is analytic on a simply connected region A and γ is a simple closed curve in A. Fix z_0 in A, but not on γ. Use the Cauchy integral formulas to show that

$$\int_\gamma \frac{f'(z)}{z-z_0}\,dz = \int_\gamma \frac{f(z)}{(z-z_0)^2}\,dz.$$

(b) Challenge: Redo part (a), but drop the assumption that A is simply connected.

Problem 10. (0 points)
Suppose $f(z)$ is entire and $\displaystyle\lim_{z\to\infty}\frac{f(z)}{z} = 0$. Show that $f(z)$ is constant.
You may use Morera's theorem: if $g(z)$ is analytic on $A - \{z_0\}$ and continuous on A, then f is analytic on A.

Problem 11. (0 points)
(a) Compute $\displaystyle\int_C \frac{\cos(z)}{z}\,dz$, where C is the unit circle.

(b) Compute $\displaystyle\int_C \frac{\sin(z)}{z}\,dz$, where C is the unit circle.

(c) Compute $\displaystyle\int_C \frac{z^2}{z-1}\,dz$, where C is the circle $|z| = 2$.

(d) Compute $\displaystyle\int_C \frac{\mathrm{e}^z}{z^2}\,dz$, where C is the circle $|z| = 1$.

(e) Compute $\displaystyle\int_C \frac{z^2-1}{z^2+1}\,dz$, where C is the circle $|z| = 2$.

(f) Compute $\displaystyle\int_C \frac{1}{z^2+z+1}\,dz$ where C is the circle $|z| = 2$.

End of pset 4