2 The zeta function

The Riemann **zeta function** is initially defined for real $s > 1$ by the convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

As in the case of the gamma function, ζ can be continued into the complex plane. There are several proofs of this fact, and we present in the next section the one that relies on the functional equation of ζ.

2.1 Functional equation and analytic continuation

In parallel to the gamma function, we first provide a simple extension of ζ to a half-plane in $\mathbb{C}$.

Proposition 2.1 *The series defining* $\zeta(s)$ *converges for* $\mathrm{Re}(s) > 1$*, and the function* ζ *is holomorphic in this half-plane.*

Proof. If $s = \sigma + it$ where σ and t are real, then

$$|n^{-s}| = |e^{-s\log n}| = e^{-\sigma \log n} = n^{-\sigma}.$$

As a consequence, if $\sigma > 1 + \delta > 1$ the series defining ζ is uniformly bounded by $\sum_{n=1}^{\infty} 1/n^{1+\delta}$, which converges. Therefore, the series $\sum 1/n^s$ converges uniformly on every half-plane $\mathrm{Re}(s) > 1 + \delta > 1$, and therefore defines a holomorphic function in $\mathrm{Re}(s) > 1$.

We shall now present a more elementary approach to the analytic continuation of the zeta function, which easily leads to its extension in the half-plane $\mathrm{Re}(s) > 0$. This method will be useful in studying the growth properties of ζ near the line $\mathrm{Re}(s) = 1$ (which will be needed in the next chapter). The idea behind it is to compare the sum $\sum_{n=1}^{\infty} n^{-s}$ with the integral $\int_1^{\infty} x^{-s}\, dx$.

Proposition 2.5 *There is a sequence of entire functions* $\{\delta_n(s)\}_{n=1}^{\infty}$ *that satisfy the estimate* $|\delta_n(s)| \leq |s|/n^{\sigma+1}$, *where* $s = \sigma + it$, *and such that*

$$\sum_{1 \leq n < N} \frac{1}{n^s} - \int_1^N \frac{dx}{x^s} = \sum_{1 \leq n < N} \delta_n(s), \tag{8}$$

whenever N *is an integer* > 1.

This proposition has the following consequence.

Corollary 2.6 *For* $\mathrm{Re}(s) > 0$ *we have*

$$\zeta(s) - \frac{1}{s-1} = H(s),$$

where $H(s) = \sum_{n=1}^{\infty} \delta_n(s)$ *is holomorphic in the half-plane* $\mathrm{Re}(s) > 0$.

To prove the proposition we compare $\sum_{1\leq n<N} n^{-s}$ with $\sum_{1\leq n<N}\int_n^{n+1} x^{-s}\,dx$, and set

$$\delta_n(s)=\int_n^{n+1}\left[\frac{1}{n^s}-\frac{1}{x^s}\right]dx. \tag{9}$$

The mean-value theorem applied to $f(x)=x^{-s}$ yields

$$\left|\frac{1}{n^s}-\frac{1}{x^s}\right|\leq\frac{|s|}{n^{\sigma+1}},\qquad\text{whenever } n\leq x\leq n+1.$$

Therefore $|\delta_n(s)|\leq|s|/n^{\sigma+1}$, and since

$$\int_1^N\frac{dx}{x^s}=\sum_{1\leq n<N}\int_n^{n+1}\frac{dx}{x^s},$$

the proposition is proved.

Turning to the corollary, we assume first that $\mathrm{Re}(s)>1$. We let N tend to infinity in formula (8) of the proposition, and observe that by the estimate $|\delta_n(s)|\leq|s|/n^{\sigma+1}$ we have the uniform convergence of the series $\sum\delta_n(s)$ (in any half-plane $\mathrm{Re}(s)\geq\delta$ when $\delta>0$). Since $\mathrm{Re}(s)>1$, the series $\sum n^{-s}$ converges to $\zeta(s)$, and this proves the assertion when $\mathrm{Re}(s)>1$. The uniform convergence also shows that $\sum\delta_n(s)$ is holomorphic when $\mathrm{Re}(s)>0$, and thus shows that $\zeta(s)$ is extendable to that half-plane, and that the identity continues to hold there.

Remark. The idea described above can be developed step by step to yield the continuation of ζ into the entire complex plane, as shown in Problems 2 and 3. Another argument giving the full analytic continuation of ζ is outlined in Exercises 15 and 16.

As an application of the proposition we can show that the growth of $\zeta(s)$ near the line $\mathrm{Re}(s)=1$ is "mild." Recall that when $\mathrm{Re}(s)>1$, we have $|\zeta(s)|\leq\sum_{n=1}^\infty n^{-\sigma}$, and so $\zeta(s)$ is bounded in any half-plane $\mathrm{Re}(s)\geq 1+\delta$, with $\delta>0$. We shall see that on the line $\mathrm{Re}(s)=1$, $|\zeta(s)|$ is majorized by $|t|^\epsilon$, for every $\epsilon>0$, and that the growth near the line is not much worse. The estimates below are not optimal. In fact, they are rather crude but suffice for what is needed later on.

Proposition 2.7 *Suppose $s=\sigma+it$ with $\sigma,t\in\mathbb{R}$. Then for each σ_0, $0\leq\sigma_0\leq 1$, and every $\epsilon>0$, there exists a constant c_ϵ so that*

(i) $|\zeta(s)|\leq c_\epsilon|t|^{1-\sigma_0+\epsilon}$, *if $\sigma_0\leq\sigma$ and $|t|\geq 1$.*